Introduction to

Algorithm Design and Analysis

[11] Graph Traversal

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In the last class ...

- **Dynamic Equivalence Relation**
- **Implementing disjoint set by Union-Find**
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing Find
    - Amortized analysis of wUnion-cFind

Course Contents

optimization problems

greedy

DP

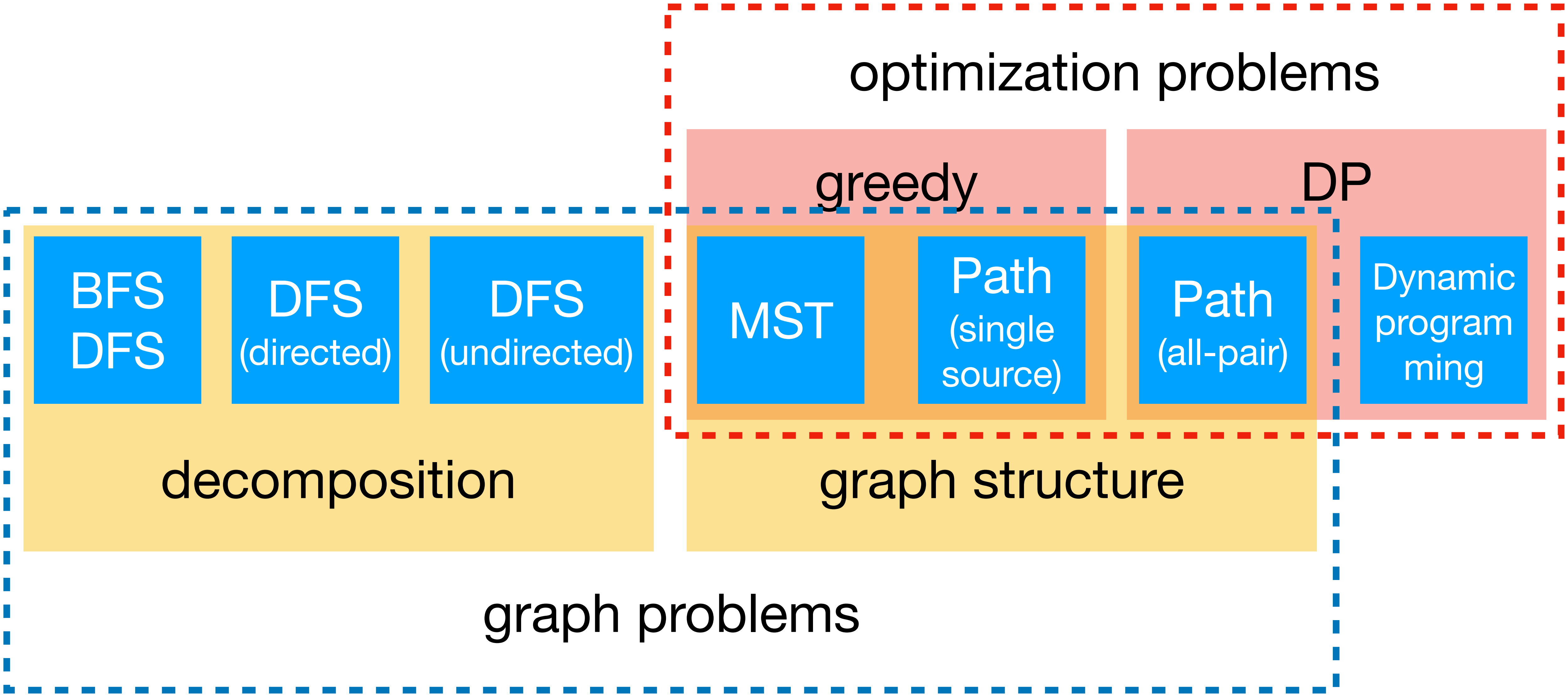
BFS DFS	DFS (directed)	DFS (undirected)	MST	Path (single source)	Path (all-pair)	Dynamic programming

decomposition

graph structure

graph problems

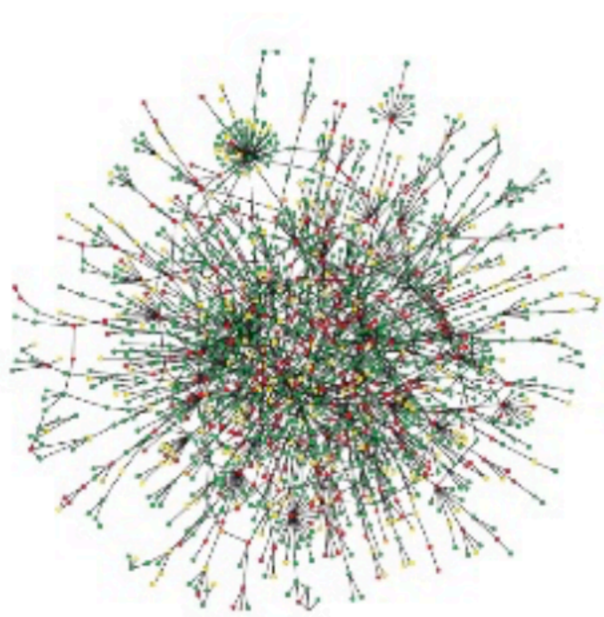
Course Contents

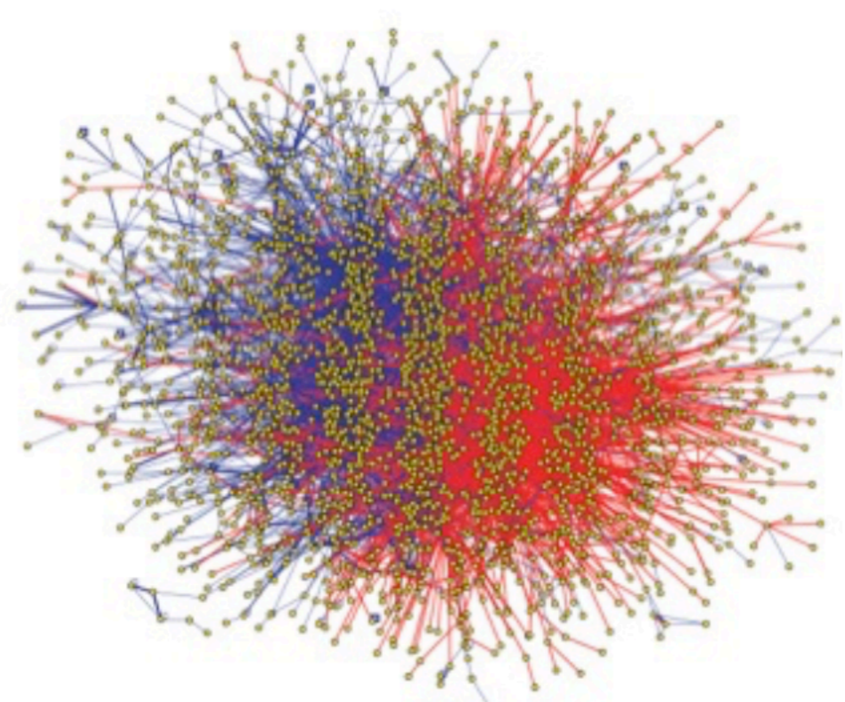


Graph Everywhere

Graph Everywhere

Protein-protein interaction network





Graph Basics

- **Node**
  - Entities of interest
  - V(G)
- **Edge**
  - Relations of interest
  - $E(G) \in V \times V$

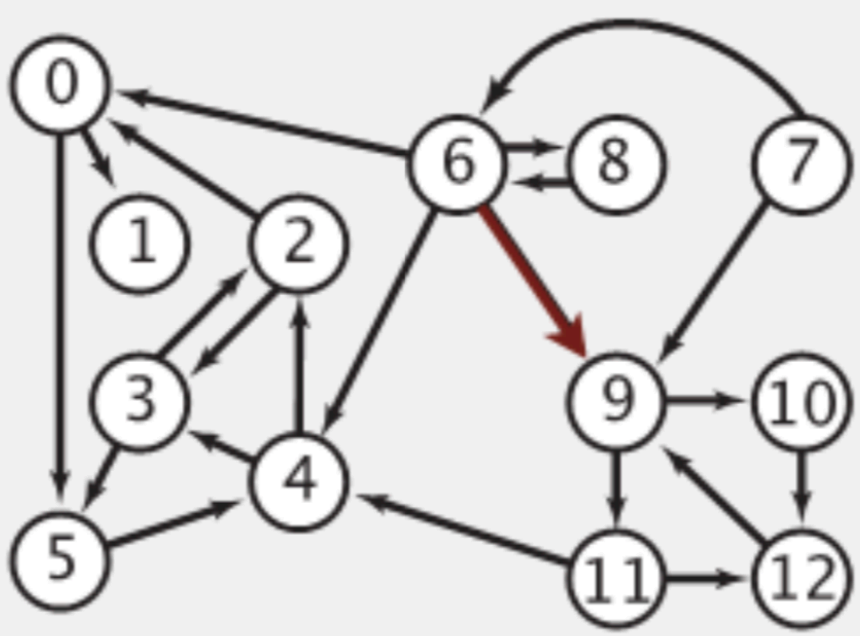
Graph Traversals

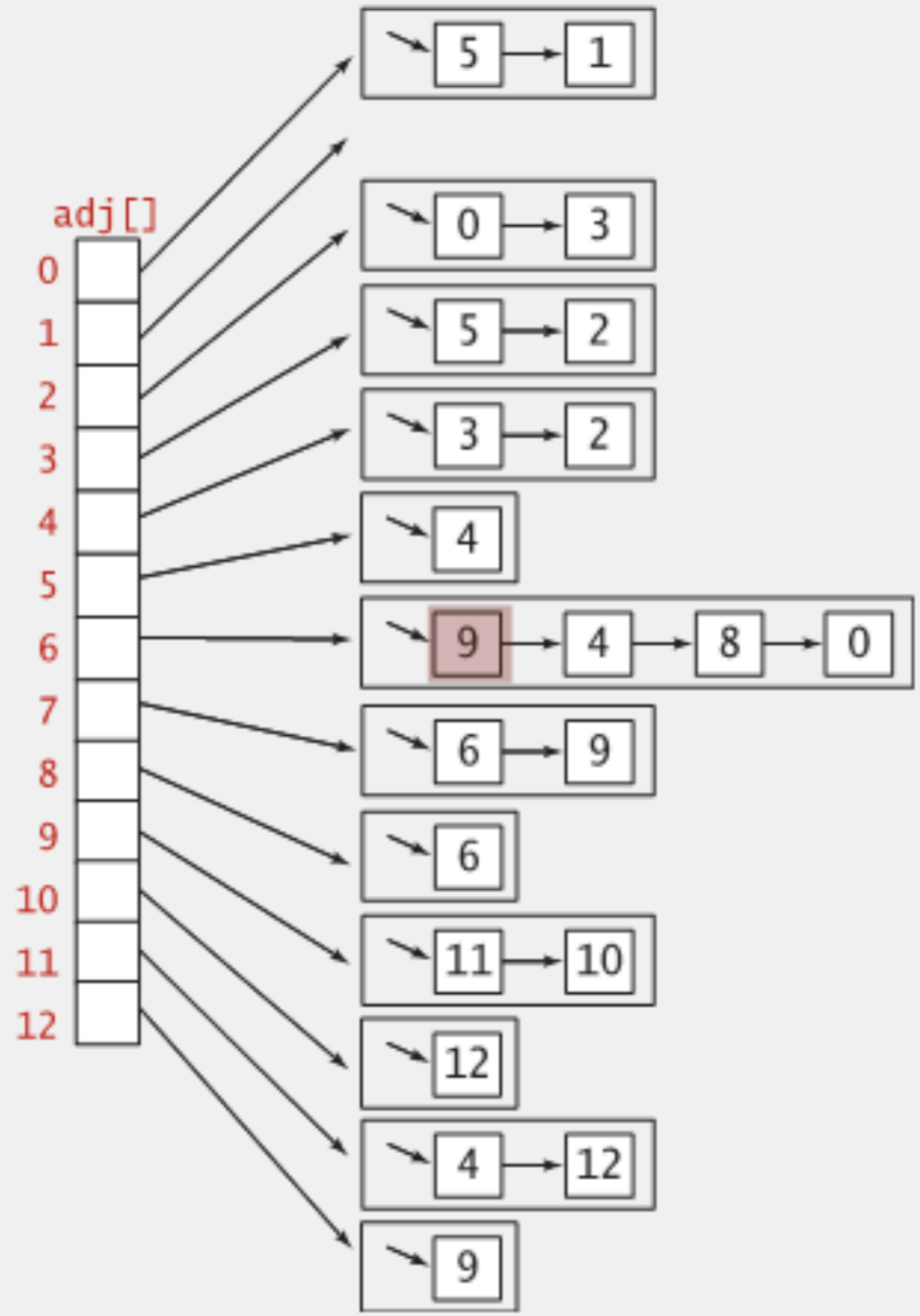
- Depth-First and Breadth-First Search
- Finding Connected Components

- General DFS/BFS Skeleton
- Depth-First Search Trace

Adjacency-lists digraph representation

Maintain vertex-indexed array of lists.



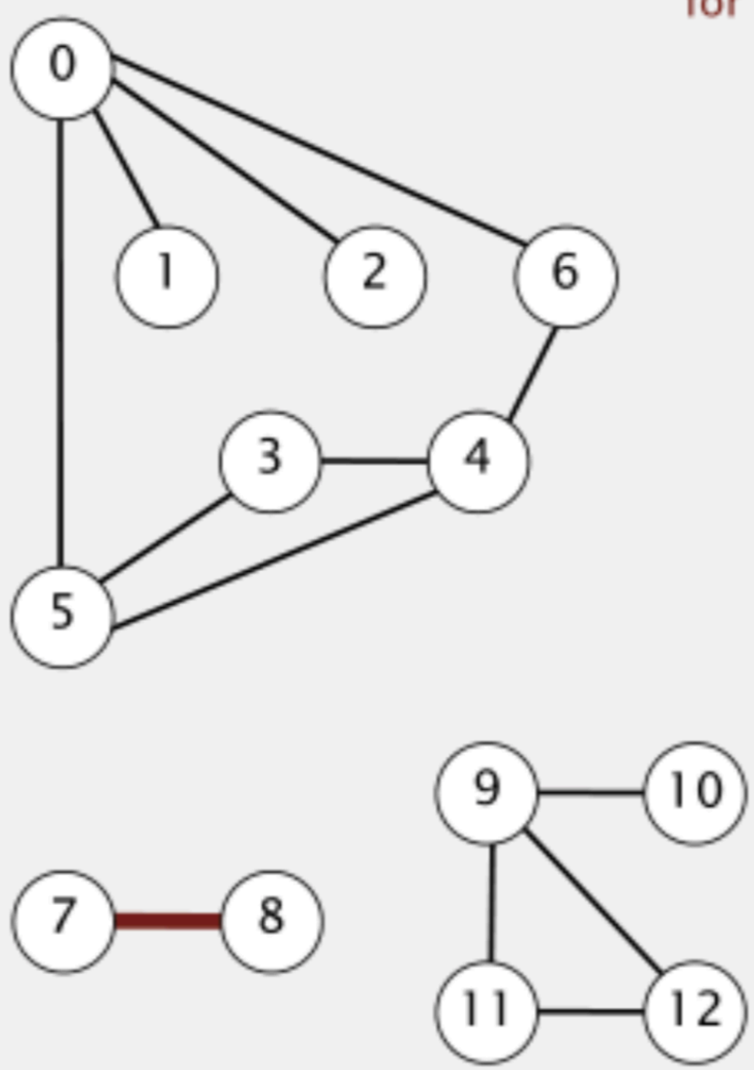


Directed vs. **Undirected** graphs

Adjacency-matrix graph representation

Maintain a two-dimensional V-by-V boolean array;

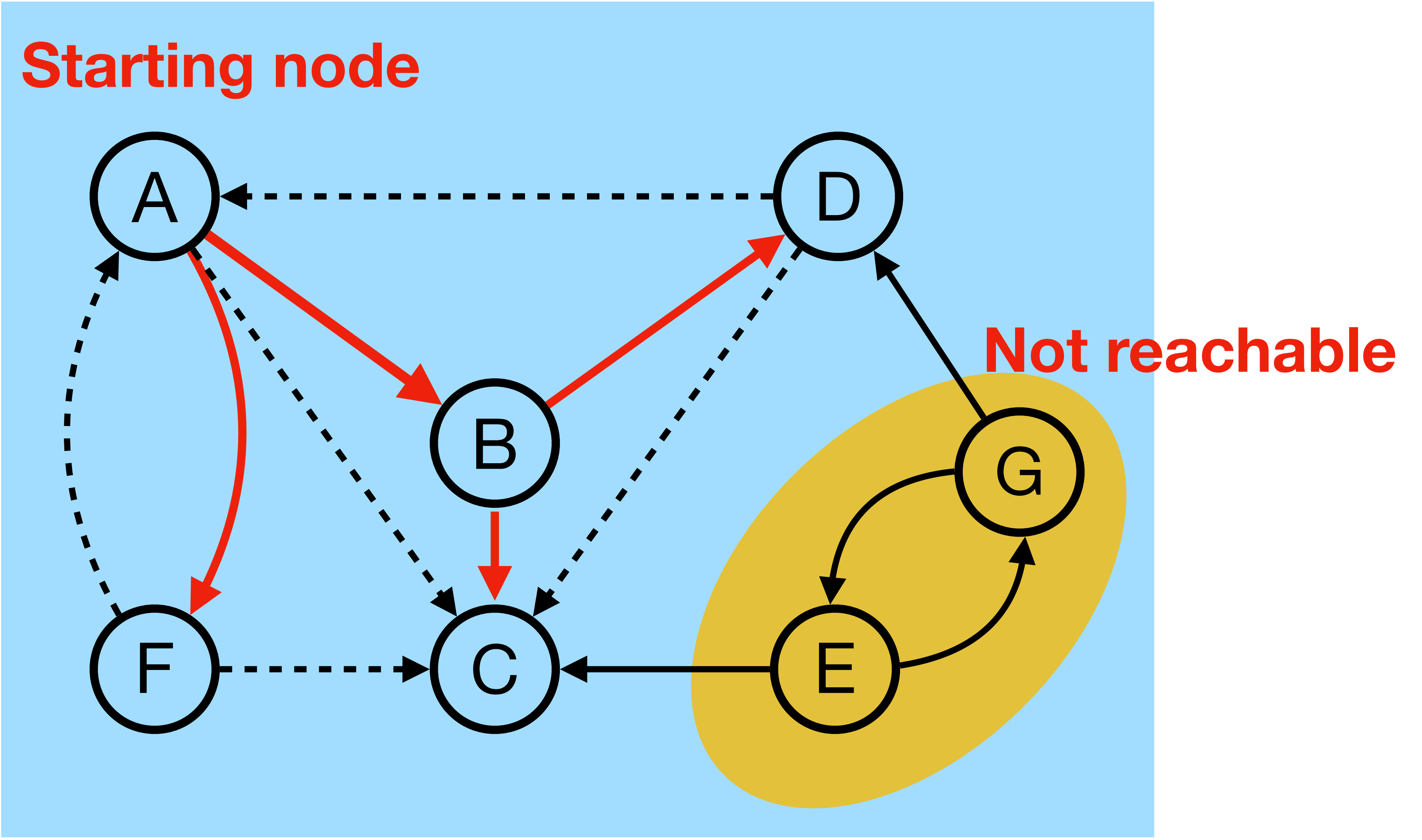
for each edge $v–w$ in graph: `adj[v][w] = adj[w][v] = true`.



two entries
for each edge

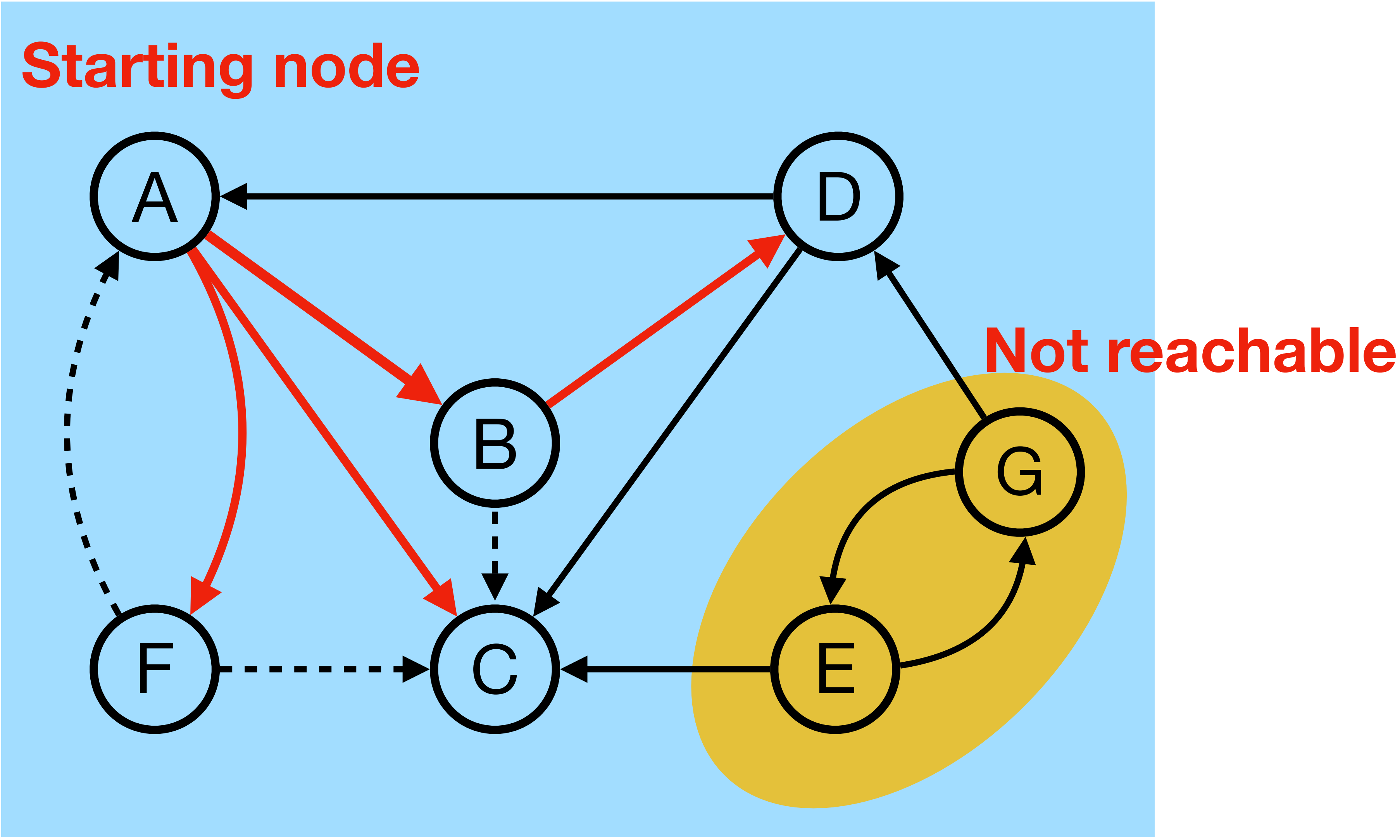
	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	1	0	0	1	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0	0	0	0
4	0	0	0	1	0	1	1	0	0	0	0	0	0
5	1	0	0	1	1	0	0	0	0	0	0	0	0
6	1	0	0	0	1	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	0	0	0	0
8	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	1	1	1
10	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	1	0	0	1
12	0	0	0	0	0	0	0	0	0	1	0	1	0

Graph Traversal



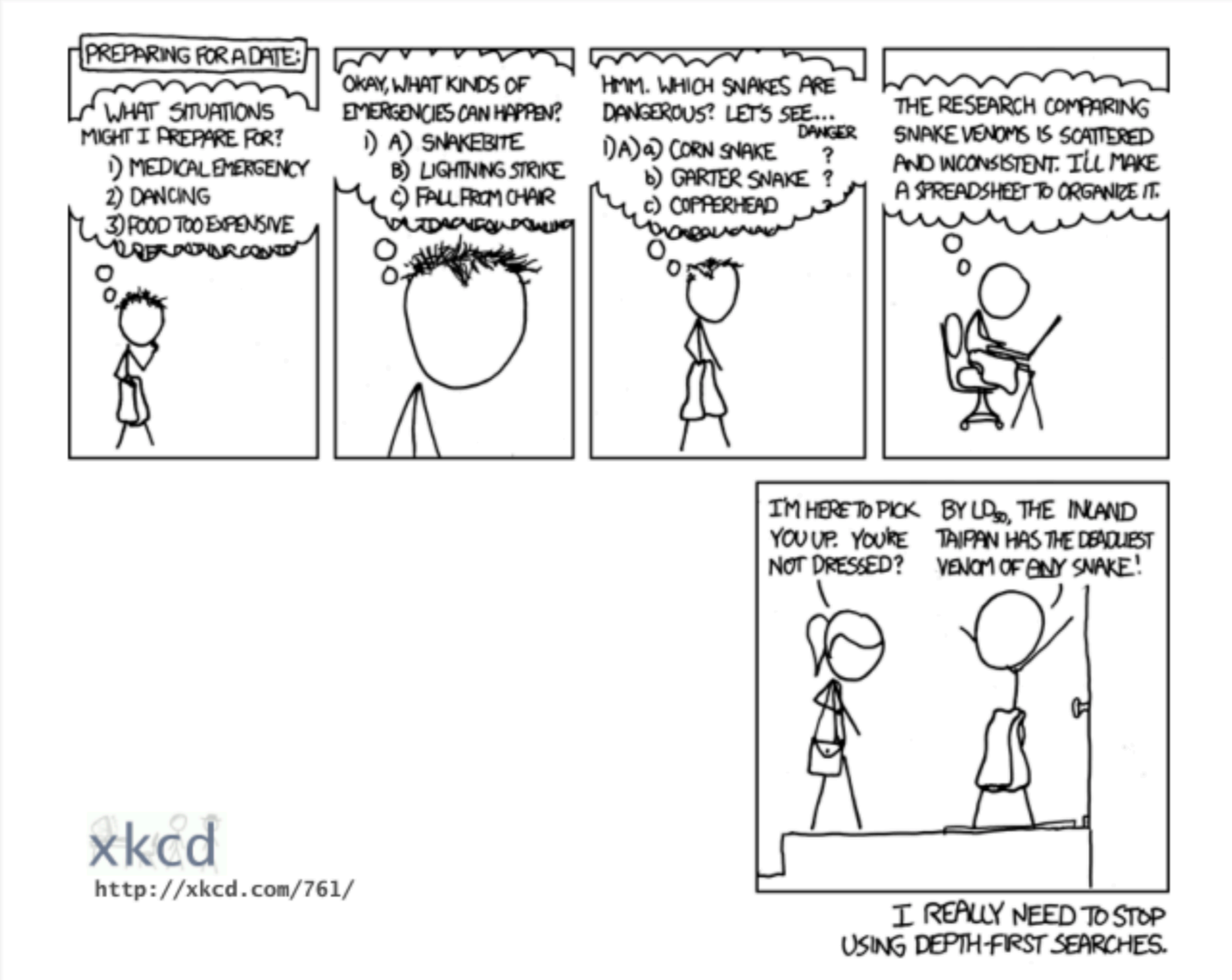
Depth-First Search

Graph Traversal



Breadth-First Search

Depth-first search application: preparing for a date



Outline of Depth-First Search

- dfs(G, v)
-     Mark v as “discovered”.
-     For each vertex w that edge vw is in G:
-         If w is undiscovered:
-             dfs(G, w)
-         Otherwise:
-             “check” vw without visiting w.
-     Mark v as “finished”.

A vertex must be exact one of three different status:
Undiscovered
Discovered but not finished
Finished

That is: exploring vw, visiting w, exploring from there as much as possible, and backtrack from w to v.

Outline of Breadth-First Search

- bfs(G, v)
- Mark s as "discovered";
- enqueue(pending, s);
- while (pending is nonempty)
- dequeue(pending, v);
- For each vertex w that edge vw is in G:
- If w is "undiscovered"
- Mark w as "discovered" and enqueue(pending, w)
- Mark v as "finished";

Finding Connected Components

- Input: a symmetric digraph G, with n nodes and 2m edges (interpreted as undirected graph), implemented as a array adjVertices[1,…n] of adjacency lists.
- Output: an array cc[1..n] of component number for each node v_i.
- void connectedComponents(intlist[] adjVertices, int n, int[] cc)// This is a wrapper procedure
- int[] color=new int[n+1];
- int v;
- <initialize color array to white for all vertices>
- for(v=1; v≤n; v++)
- if(color[v]==white)
- ccDFS(adjVertices, color, v, v, cc);
- return

Depth-first search

ccDFS: the procedure

- void ccDFS(intList[] adjVertices, int[] color, int v, int ccNum, int[] cc)// v as the code of current connected component
- int w;
- intList remAdj;
- color[v]=gray;
- cc[v]=ccNum;
- remAdj=adjVertices[v];
- while(remAdj != nil)
- w=first(remAdj);
- if(color[w]==white)
- ccDFS(adjVertices, color, w, ccNum, cc);
- remAdj=rest(remAdj);
- color[v]=black;
- return

The elements of *remAdj* are neighbors of *v*

Processing the next neighbor, if existing, another depth-first search to be incurred

v finished

Analysis of CC Algorithm

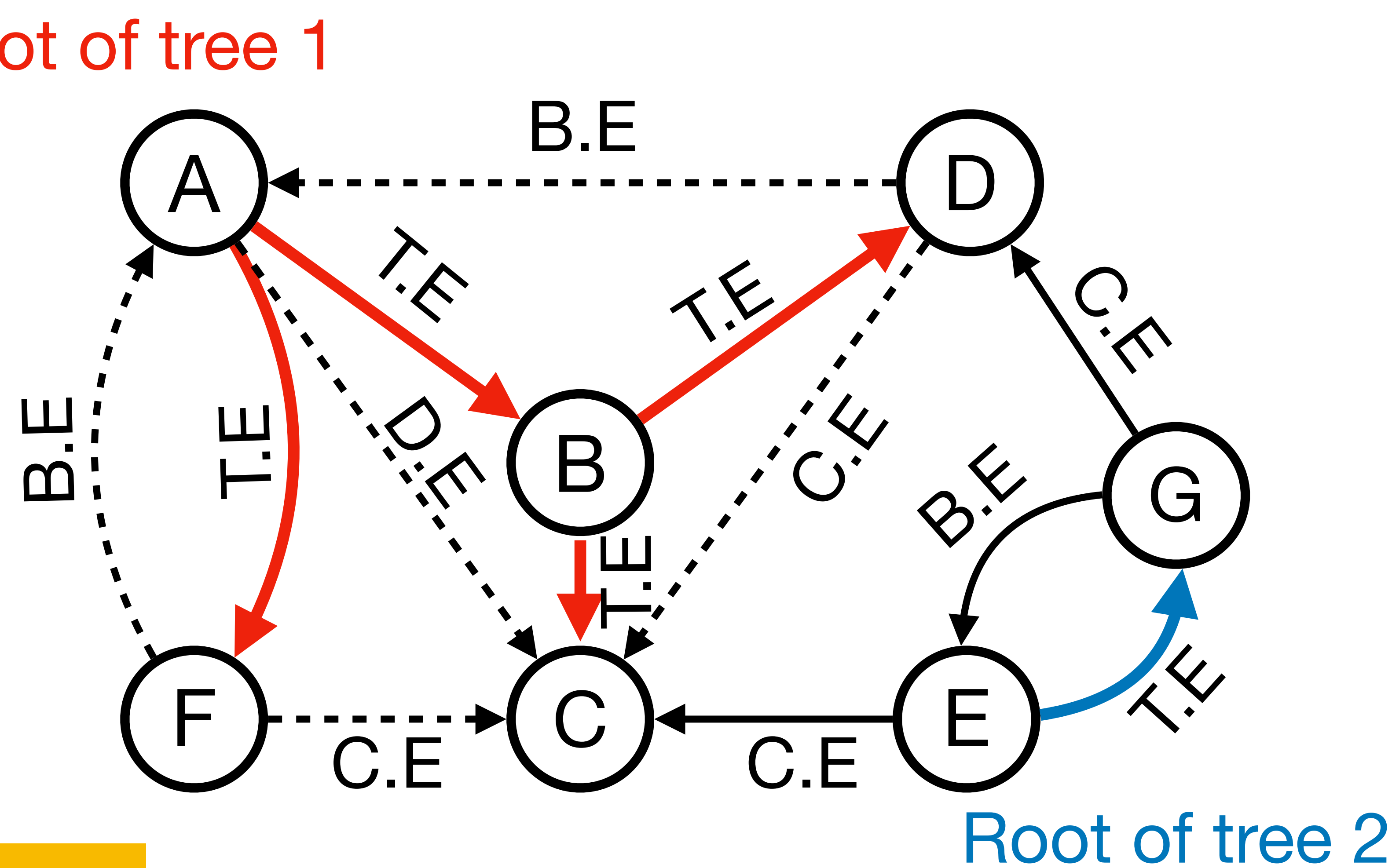
- **connectedComponents, the wrapper**
  - Linear in n (color array initialization+for loop on adjVertices)
- **ccDFS, the depth-first searcher**
  - In one execution of ccDFS on v, the number of instructions(rest(remAdj)) executed is proportional to the size of adjVertices[v].
  - Note: ∑ (size of adjVertices[v]) is 2m, and the adjacency lists are traversed **only once**.
- **So, the time complexity is in Θ(m+n)**
  - Extra space requirements:
    - Color array
    - Activation frame stack for recursion

Visits On a Vertex

- **Classification for visits on a vertex**
  - First visit (exploring): status: white->gray
  - (Possibly) multi-visits by backtracking to: status keeps gray
  - Last visit (no more branch-finished): status: gray->black
- **Different operations can be done, during the different visits on a specific vertex**
  - On the vertex
  - On (selected) incident edges

Depth-first Search Trees

DFS forest={(DFS tree1), (DFS tree2)}



T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

A finished vertex is never revisited, such as C.

Depth-First Search — Generalized Skeleton

- Input: Array adjVertices for graph G
- Output: Return value depends on application
- int dfsSweep(intList[], adjVertices, int n, …)
-     int ans;
-     <Allocate color array and initialize to white>
-     for each vertex v of G, in some order
-         if(color[v]==white)
-             int vAns=dfs(adjVertices, color, v, …);
-             <Process vAns>
-         // continue loop
-     return ans;

Depth-First Search — Generalized Skeleton

- int dfs(intList[] adjVertices, int[] color, int v, …)
- int w;
- intList remAdj;
- int ans;
- color[v]=gray;
- <Preorder processing of vertex v>
- remAdj=adjVertices[v];
- while(remAdj != nil)
- w=first(remAdj);
- if(color[w]==white)
- <Exploratory processing for tree edge vw>
- int wAns=dfs(adjVertices, color, w, …)
- <Backtrack processing for tree edge vw, using wAns>
- else
- <Checking for nontree edge vw>
- remAdj=rest(remAdj);
- <Postorder processing of vertex v, including final computation of ans>
- color[v]=black;
- return ans;

If partial search is used for a application, tests for termination may be inserted here.

Specialized for connected components:
Parameter added
Preorder processing inserted - cc[v] =ccNum

Breadth-First Search — Skeleton

- Input: Array adjVertices for graph G
- Output: Return value depends on application
- void bfsSweep(intList[], adjVertices, int n, …)
- int ans;
- <Allocate color array and initialize to white>
- for each vertex v of G, in some order
- if(color[v]==white)
- void bfs(adjVertices, color, v, …);
- // continue loop
- return;

Breadth-First Search — Skeleton

- void bfs(intList[] adjVertices, int[] color, int v, …)
- int w; intList remAdj; Queue pending;
- color[v]=gray; enqueue(pending, v);
- while(pending is nonempty)
- w=dequeue(pending); remAdj=adjVertices[w];
- while(remAdj!=nil)
- x=first(remAdj);
- if(color[x]==while)
- color[x]=gray; enqueue(pending, x);
- remAdj=rest(remAdj);
- <processing of vertex w>
- color[w]=black;
- return

Can be further generalized

DFS v.s. BFS Search

- **Processing opportunities for a node**
  - Depth-first: 2
    - At discovering
    - At finishing
  - Breadth-first: only 1, when de-queued
  - At the second processing opportunity for the DFS, the algorithm can make use of information about the descendants of the current node.

Time Relation on Changing Color

- **Keeping the order in which vertices are encountered for the first or last time**
  - A global integer time: 0 as the initial value, incremented with each color changing for *any* vertex, and the final value is 2*n*
  - Array *discoverTime*: the *i* th element records the time vertex v_i turns into gray
  - Array *finishTime*: the *i* th element records the time vertex v_i turns into black
  - The active interval for vertex *v,* denoted as *active*(*v*), is the duration while *v* is gray, that is:

discoverTime[v], …, finishTime[v]

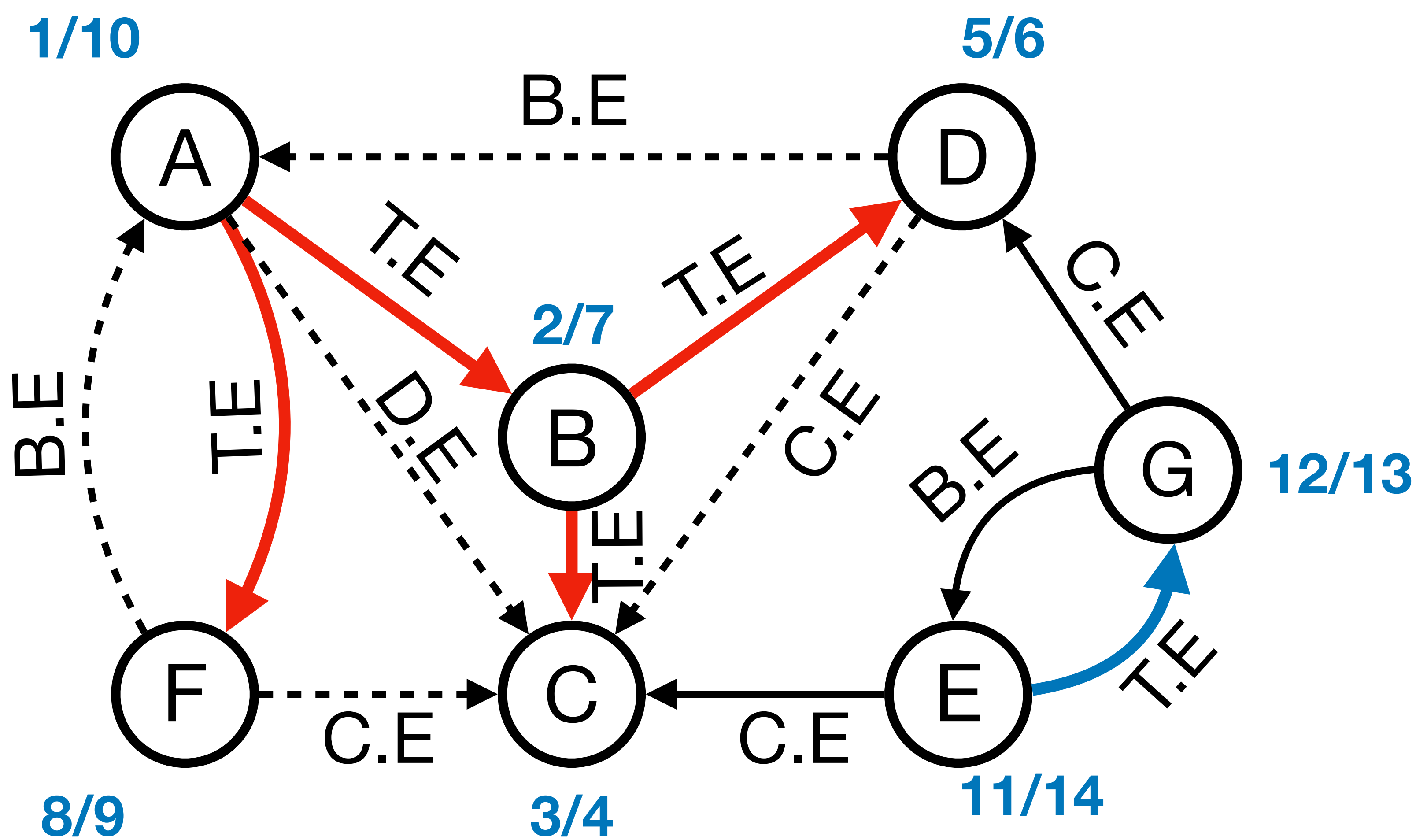
Depth-First Search Trace

- General DFS skeleton modified to compute discovery and finishing times and “construct” the depth-first search forest.
- int dfsTraceSweep(intList[] adjVertices, int n, int[] discoverTime, int[] finishTime, int[] parent)
- int ans; int time=0;
- <Allocate color array and initialize to white>
- for each vertex v of G, in some order
- if(color[v]==white)
- parent[v]=-1
- int vAns=dfsTrace(adjVertices, color, v, discoverTime, finishTime, parent, time);
- //continue loop
- return ans;

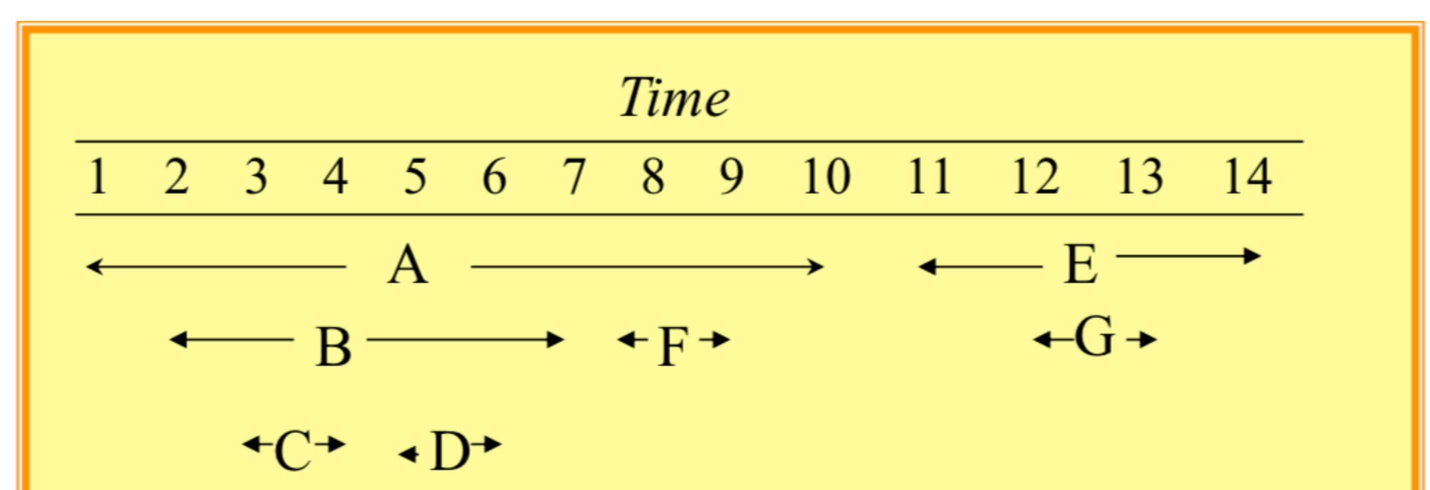
Depth-First Search Trace

- int dfsTrace(intList[] adjVertices, int[] color, int v, int[] discoverTime, int[] finishTime, int[] parent, int time)
- int w; intList remAdj; int ans;
- color[v]=gray; time++; discoverTime[v]=time;
- remAdj=adjVertices[v];
- while(remAdj != nil)
- w=first(remAdj);
- if(color[w]==white)
- parent[w]=v;
- Int wAns=dfs(adjVertices, color, w, discoverTime, finishTime, parent, time);
- else <Checking for nontree edge vw>
- remAdj=rest(remAdj);
- time++; finishTime[v]=time;
- color[v]=black;
- Return ans;

Active Interval



The relations are summarized in the next frame



T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

Properties of Active Intervals(1)

- **If *w* is a descendant of *v* in the DFS forest, then *active*(*w*)$\subseteq$*active*(*v*), and the inclusion is proper if $w \neq v$.**
- **Proof:**
  - Define a partial order $<$: $w<v$ iff. w is a proper descendants of v in its DFS tree. The proof is by induction on $<$.
  - If v is minimal. The only descendant of v is itself. Trivial.
  - Assume that for all $x<v$, if w is a descendant of x, then active(w)$\subseteq$active(x).
  - Let w be any proper descendant of v in the DFS tree, there must be some x such that vx is a tree edge on the tree path to w, so w is a descendant of x. According to dfsTrace, we have *active*(*x*)$\subset$*active*(*v*), by inductive hypothesis, *active*(*w*)$\subset$*active*(*v*).

Properties of Active Intervals(2)

- **If *active*(*w*)⊆*active*(*v*), then *w* is a descendant of *v*. And if *active*(*w*)⊂*active*(*v*), then *w* is a proper descendant of *v*.**
- **That is: *w* is discovered while *v* is active.**
- **Proof:**
  - If w is **not** a descendant of v, there are two cases:
    - v is a proper descendant of w, then *active*(*v*)⊂*active*(*w*), so, it is impossible that *active*(*w*)⊆*active*(*v*), contradiction.
    - There is no ancestor/descendant relationship between v and w, then *active*(*w*) and *active*(*v*) are disjoint, contradiction.

Properties of Active Intervals(3)

- **If v and w have no ancestor/descendant relationship in the DFS forest, then their active intervals are disjoint.**

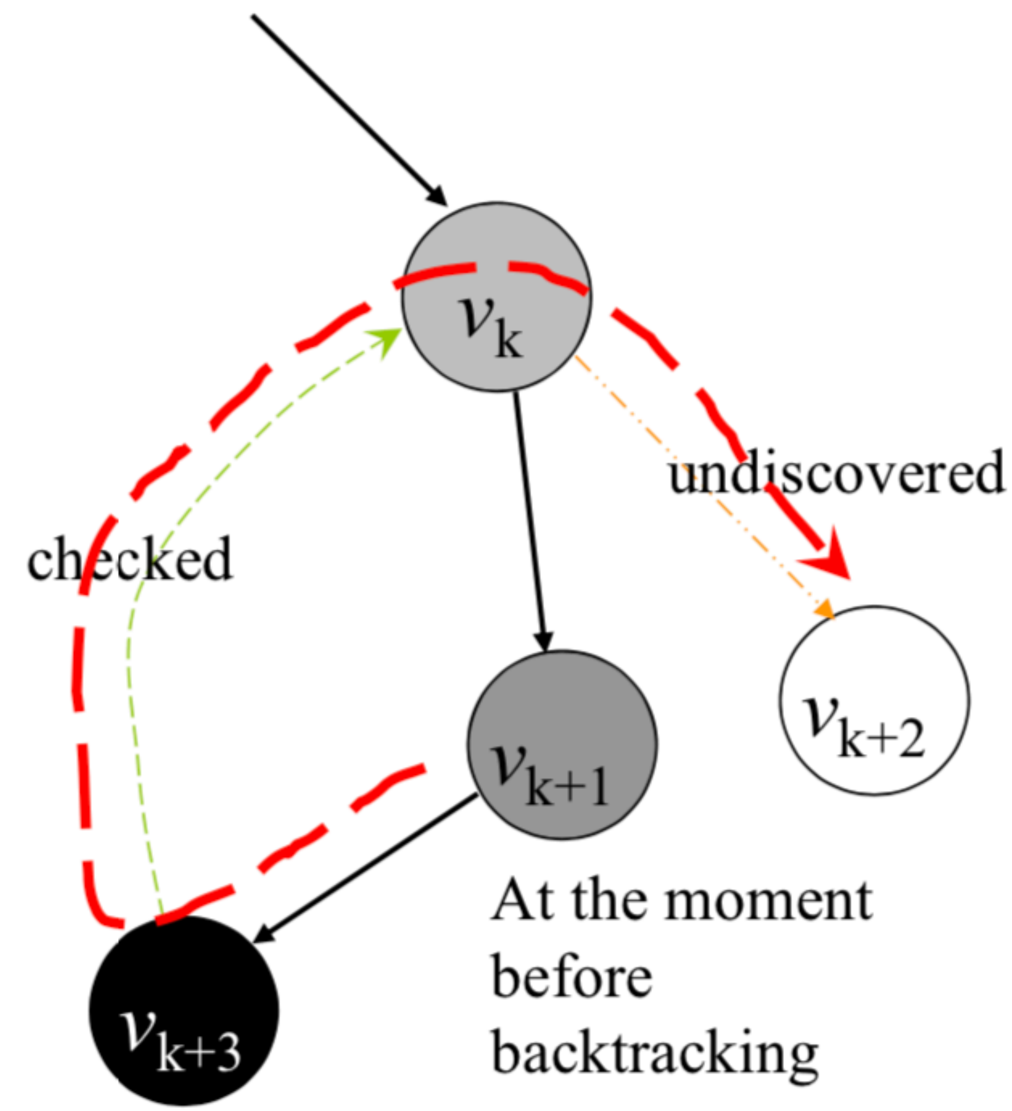
- **Proof:**
  - If v and w are in different DFS tree, it is trivially true, since the trees are processed one by one.
  - Otherwise, there must be a vertex c, satisfying that there are tree paths c to v, and c to w, without edges in common. Let the leading edges of the two tree path are cy, cz, respectively. According to dfsTrace, *active*(*y*) and *active*(*z*) are disjoint.
  - We have *active*(*v*)∈*active*(*y*), *active*(*w*)∈*active*(*z*). So, *active*(*v*) and *active*(w) are disjoint.

Properties of Active Intervals(4)

- **If edge vw $\in E_G$, then**
  - vw is a **cross edge** iff. *active*(w) entirely precedes *active*(v).
  - vw is a **descendant edge** iff. there is some third vertex *x*, such that *active*(w)⊂*active*(*x*)⊂*active*(*v*),
  - vw is a **tree edge** iff. *active*(w)⊂*active*(*v*), and there is no third vertex *x*, such that *active*(w)⊂*active*(*x*)⊂*active*(*v*),
  - vw is a **back edge** iff. *active*(v)⊂*active*(w),

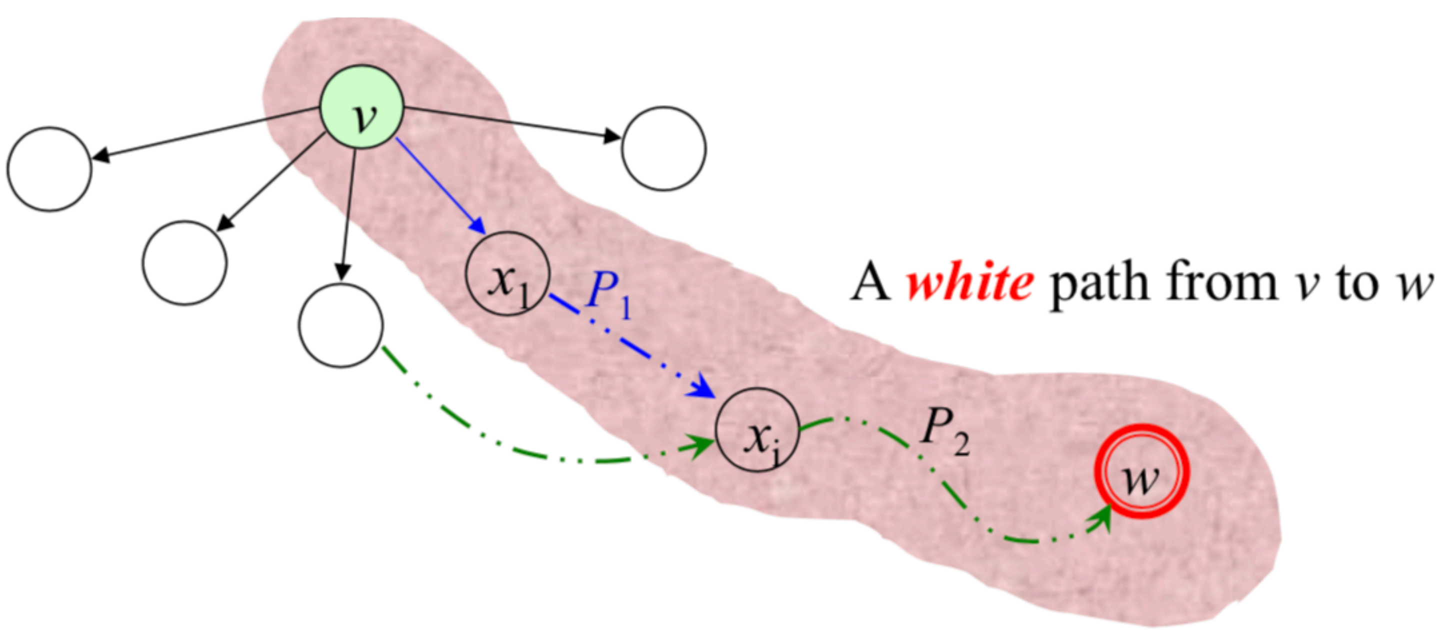
Ancestor and Descendant

- That *w* is a descendant of *v* in the DFS forest means that there is a direct path from *v* to *w* in some DFS tree.
- The path is also a path in *G*.
- However, if there is a direct path from *v* to *w* in G, is *w* necessarily a descendant of *v* in *the* DFS forest?



DFS Tree Path

- [White Path Theorem] w is a descendant of v in a DFS tree iff. At the time v is discovered (just to be changing color into gray), there is a path in G from v to w consisting entirely of white vertices.



Proof of White Path Theorem

- **Proof**
  - => all the vertices in the path are descendants of v.
  - <= by induction on the length k of a white path from v to w.
    - When k=0, v=w.
    - For k>0, let P=(v, x_1,x_2...x_k=w). There must be some vertex on P which is discovered during the active interval of v, e.g. x_1, Let x_i is earliest discovered among them. Divide P into P_1 from v to x_i, and P_2 from x_i to w. P_2 is a white path with length less than k, so, by inductive hypothesis, w is a descendant of x_i. Note: *active*(x_i)⊆*active*(v), so x_i is a descendant of v. By transitivity, w is a descendant of v.

Thank you!
Q & A