Introduction to

# Algorithm Design and Analysis

[12] Directed Acyclic Graph

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# In the last class …

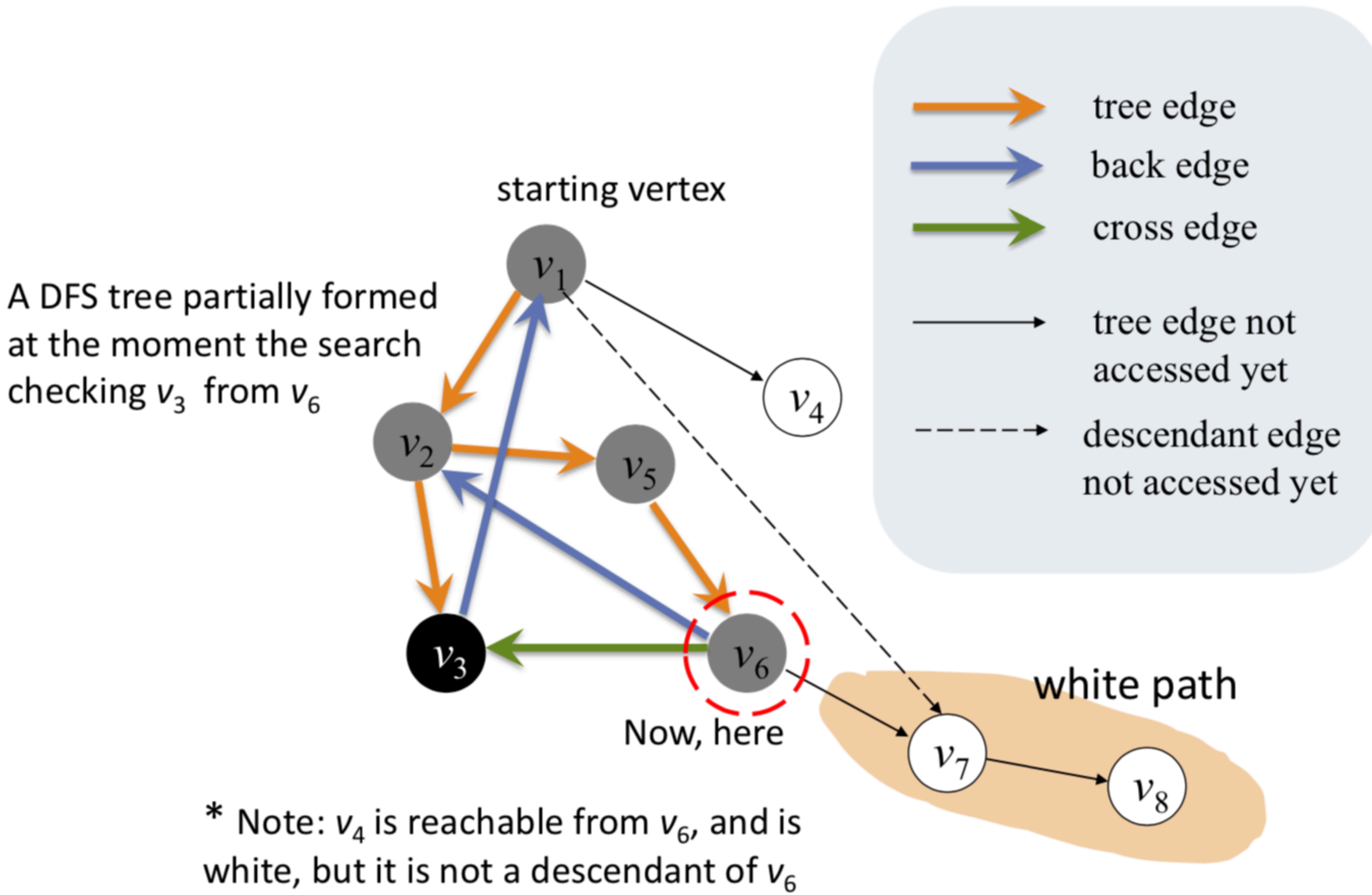
- Depth-first and breadth-first search
- Finding connected components

- General DFS/BFS skeleton
- Depth-first search trace

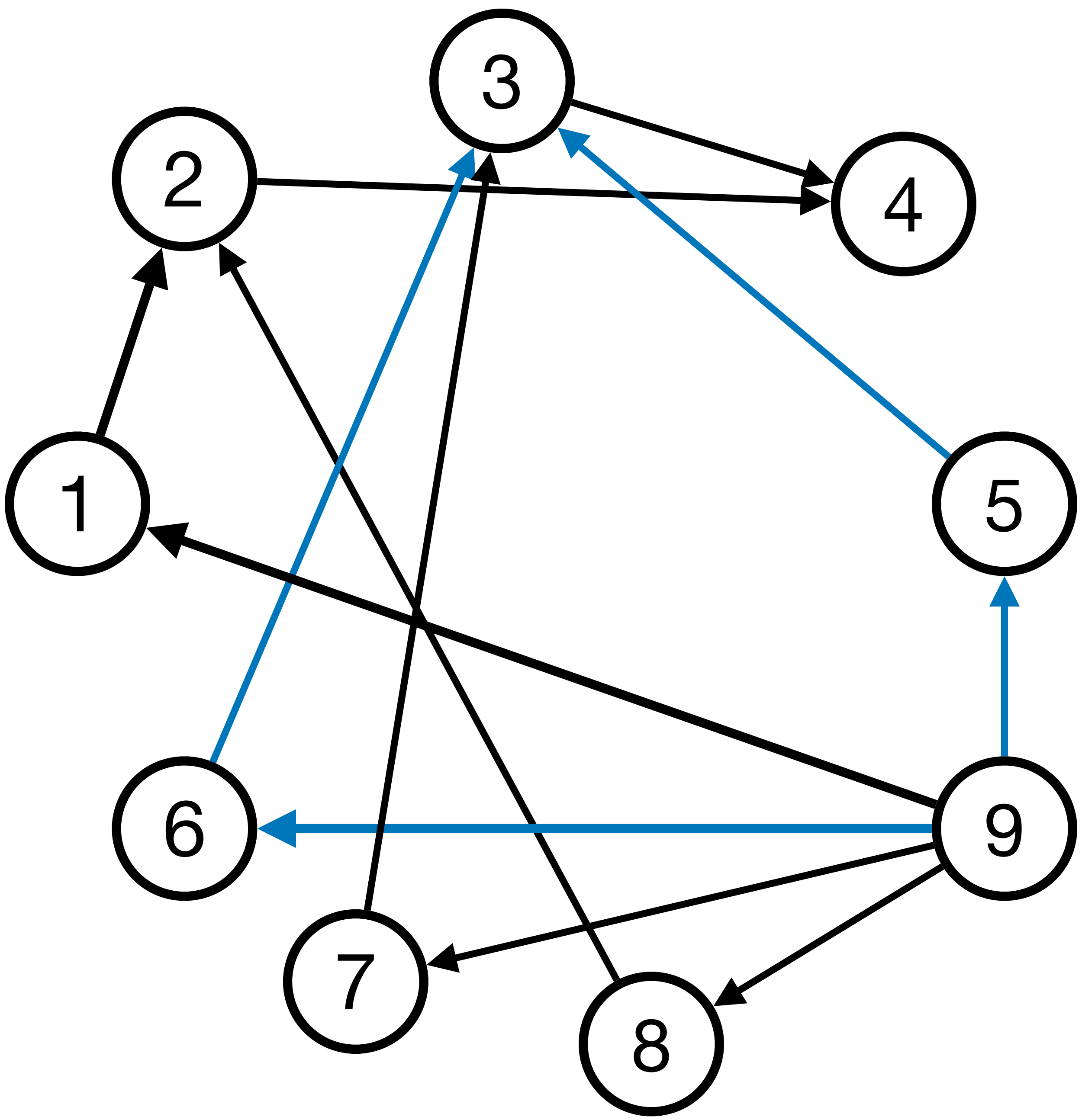
# Applications of Graph Decomposition

- **Directed Acyclic Graph**
  - Topological order
  - Critical path analysis
- **Strongly Connected Component (SCC)**
  - Strong connected component and condensation
  - The algorithm
  - Leader of strong connected component

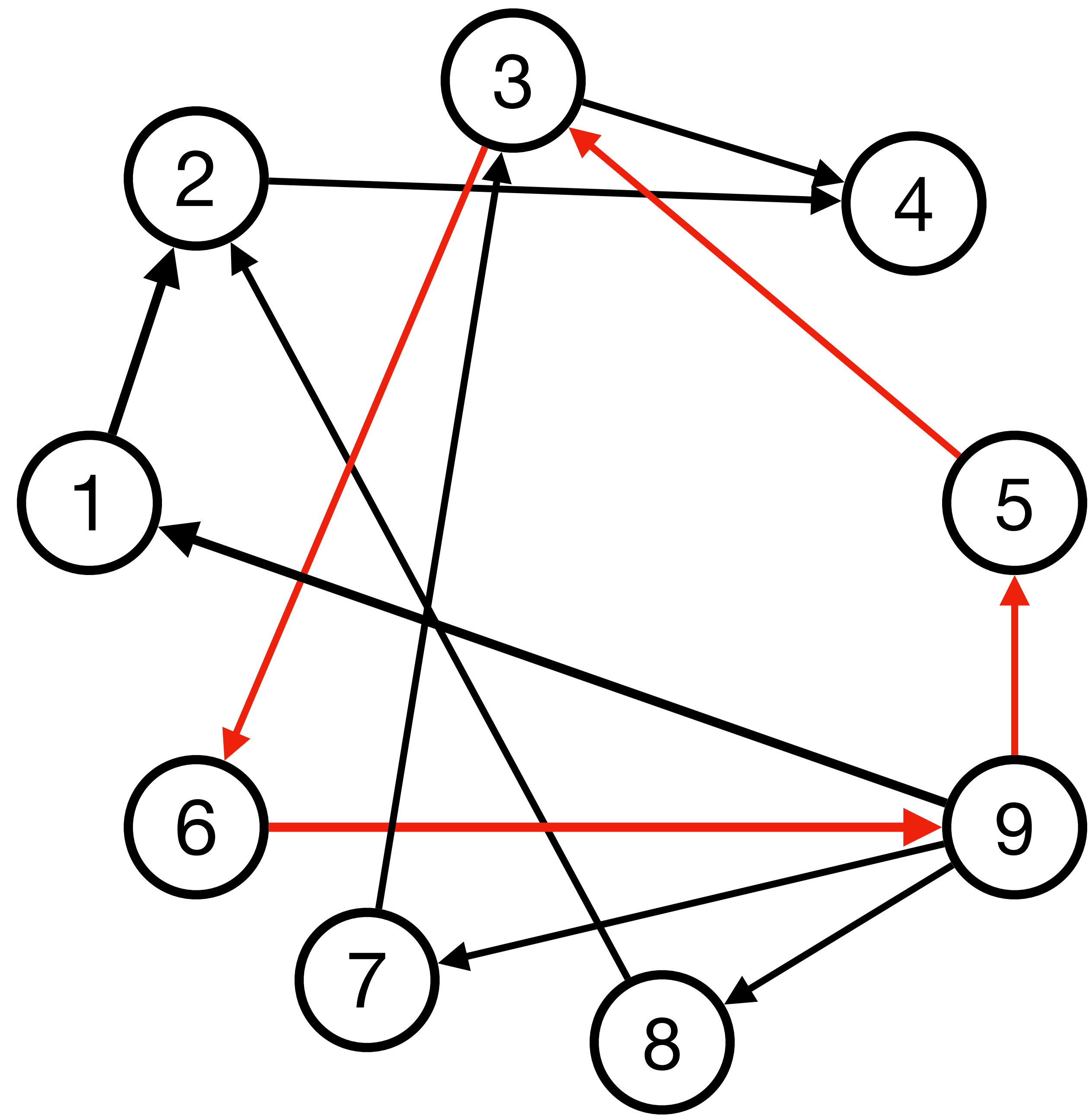
# For Your Reference



# Directed Acyclic Graph (DAG)



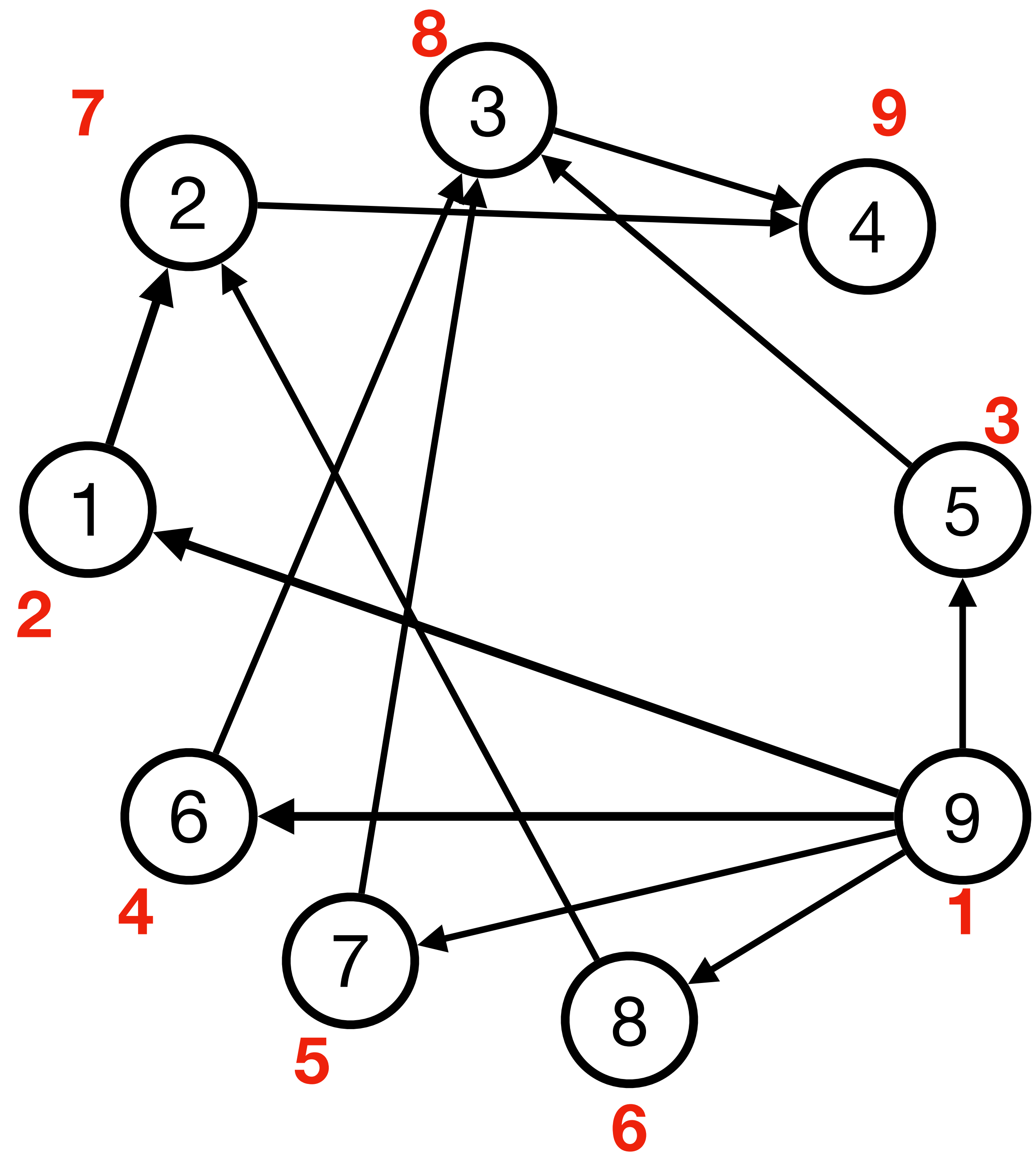

A Directed Acyclic Graph




Not a DAG

# Topological Order for G=(V, E)

- **Topological number**
  - An assignment of distinct integer 1,2,…,n to the vertices of V
  - For every vw∈E, the topological number of v is less than that of w.
- **Reverse topological order**
  - Defined similarly ("greater than")



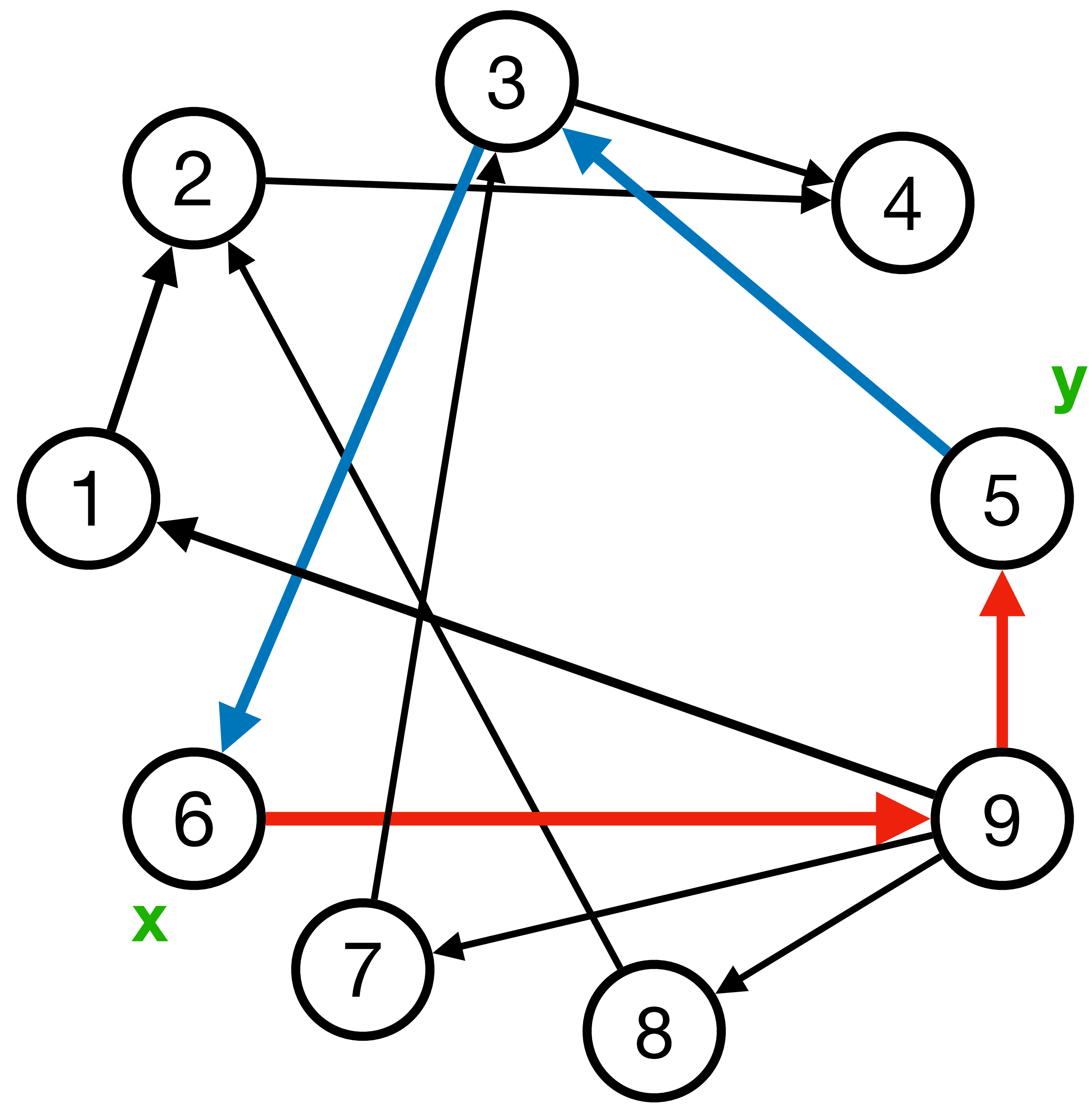
# Existence of Topological Order - a Negative Result

- **If a directed graph G has a cycle, then G has no topological order**
- **Proof**
  - [By contradiction]

yx-path

xy-path

**For any given topological order, all the vertices on both paths must be in increasing order. Contradiction results for any assignments for x and y.**



# Reverse Topological Ordering

- **Specialized parameters**
  - Array topo, keeps the topological number assigned to each vertex.
  - Counter topoNum to provide the integer to be used for topological number assignments
- **Output**
  - Array topo as filled.

# Reverse Topological Ordering

- void dfsTopoSweep(intList[] adjVertices, int n, int[] topo)
- int topoNum=0;
- <Allocate color array and initialize to white>
- for each vertex v of G, in some order
- if(color[v]==white)
- dfsTopo(adjVertices, color, v, topo, topoNum);
- //continue loop
- return;

For non-reverse topological ordering, initialized as n+1

# Reverse Topological Ordering

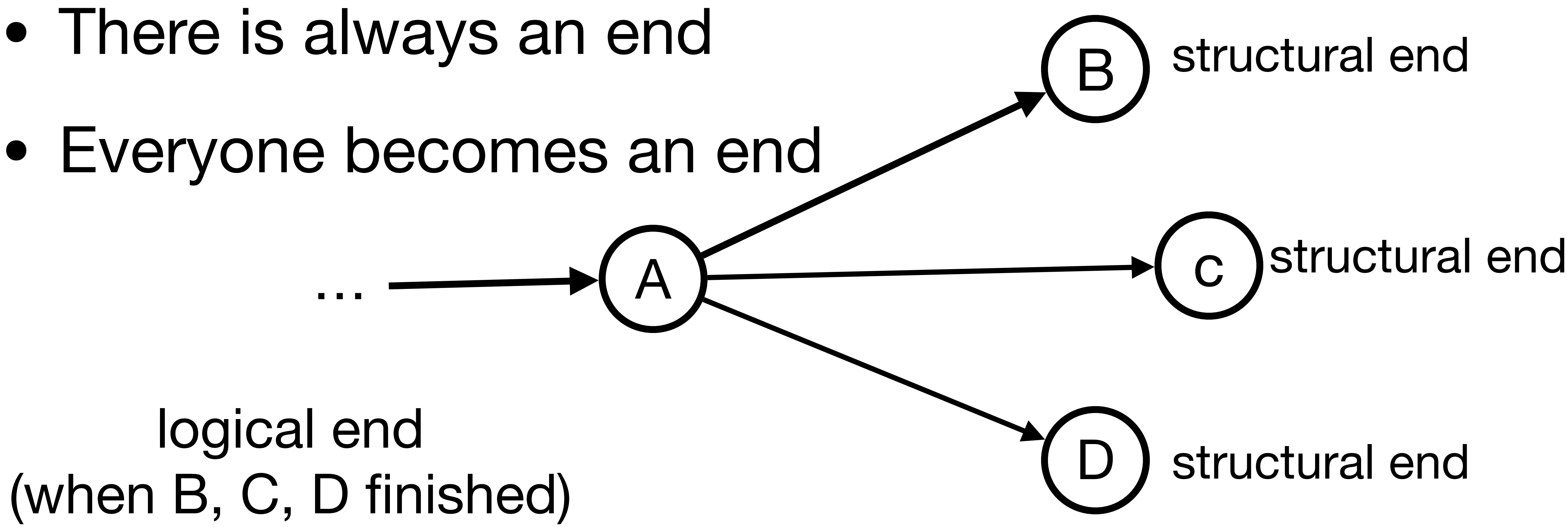
- int dfsTopo(intList[] adjVertices, int[] color, int v, int[] topo, int topoNum)
- int w; intList remAdj; color[v]=gray;
- remAdj=adjVertices[v];
- while(remAdj != nil)
- w=first(remAdj);
- if(color[w]==white)
- dfsTopo(adjVertices, color, w, topo, topoNum);
- remAdj=rest(remAdj);
- topoNum++; topo[v]=topoNum;
- color[v]=black;
- return;

Obviously, in Θ(m+n)

Filling topo is a post-order processing, so, the earlier discovered vertex has relatively greater top number

# Reverse Topological Ordering

- **For an "end node"**
  - Easy to decide
- **Acyclic**
  - There is always an end
  - Everyone becomes an end



# Correctness of the Algorithm

- **If G is a DAG with *n* vertices, the procedure *dfsTopoSweep* computes a reverse topological order for G in the array *topo*.**

- **Proof**

    - The procedure dfsTopo is called exactly once for a vertex, so, the numbers in *topo* must be distinct in the range 1,2,...*n*.

    - For any edge vw, vw can't be a back edge (otherwise, a cycle is formed). For any other edge types, we have *finishTime*(v)>*finishTime*(w), so, *topo*(w) is assigned earlier than *topo*(v). Note that *topoNum* is incremented monotonically, so, *topo*(v)>*topo*(w).

# Existence of Topological Order

- In fact, the proof of correctness of topological ordering has proved that: DAG always has a topological order.

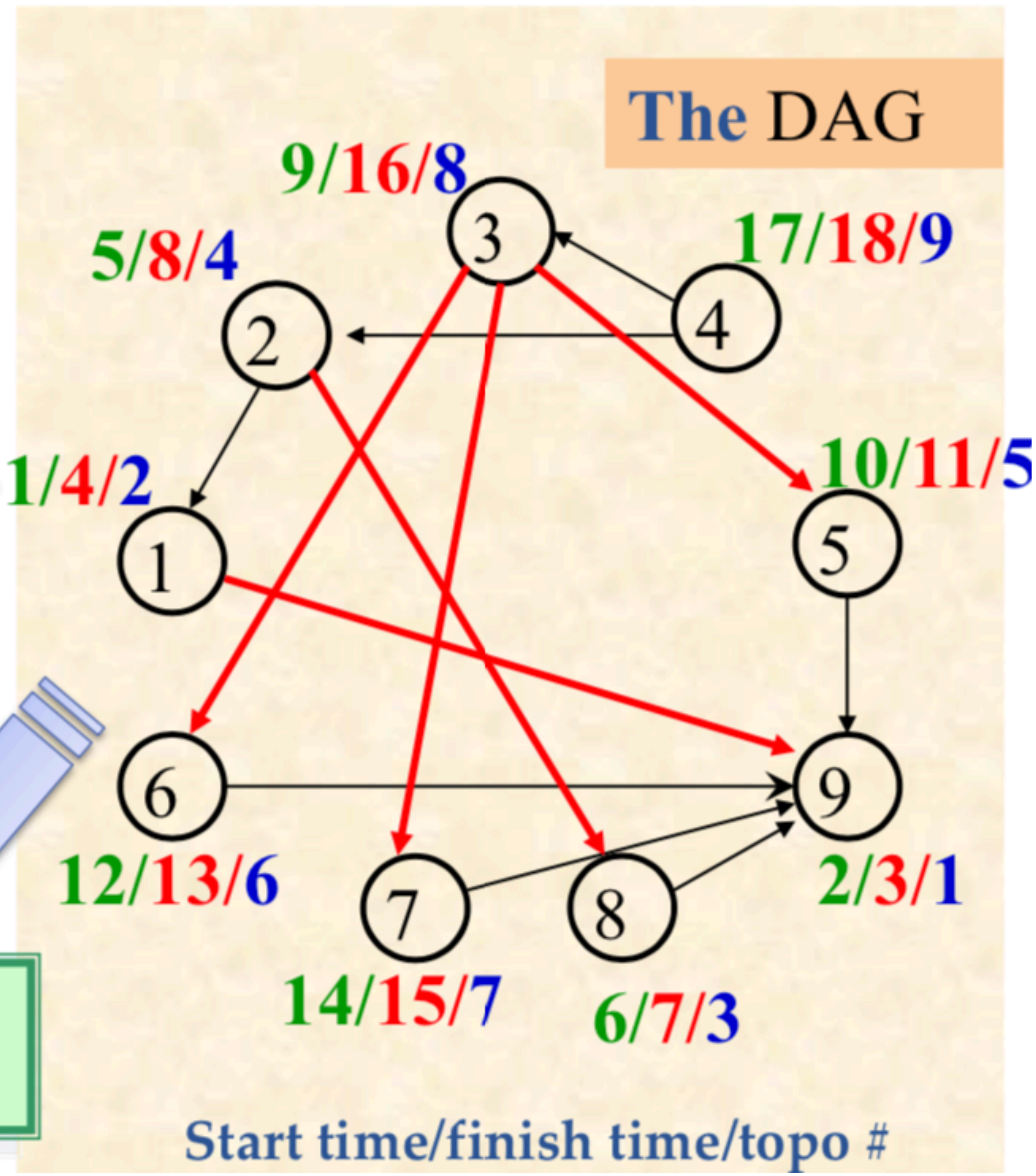
- So, G has a topological ordering, iff. G is a directed acyclic graph.

# Task Scheduling

- **Problem:**
  - Scheduling a project consisting of a set of interdependent tasks to be done by one person.
- **Solution:**
  - Establishing a dependency graph, the vertices are tasks, and edge vw is included iff. the execution of v depends on the completion of w,
  - Making task scheduling according to the topological order of the graph (if existing).

# Task Scheduling: an Example

| Tasks(No.) | Depends on |
|---|---|
| choose clothes(1) | 9 |
| dress(2) | 1,8 |
| eat breakfast(3) | 5,6,7 |
| leave(4) | 2,3 |
| make coffee(5) | 9 |
| make toast(6) | 9 |
| pour juice(7) | 9 |
| shower(8) | 9 |
| wake up(9) | - |




A reverse topological order

**9, 1, 8, 2, 5, 6, 7, 3, 4**

# Project Optimization Problem

Assuming that parallel executions of tasks ($v_i$) are possible except for prohibited by interdependency.

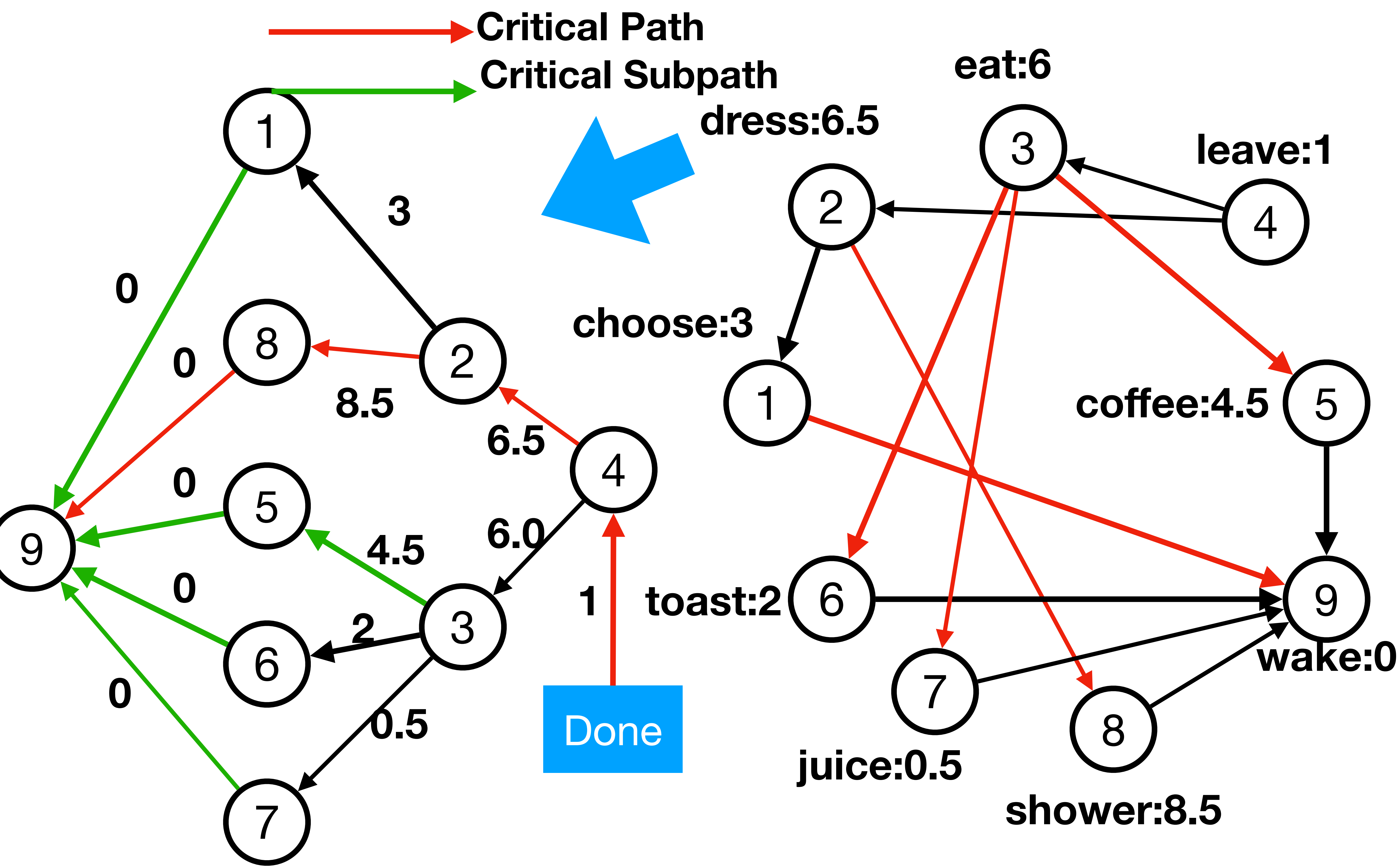
- **Observation**
  - In a critical path, $v_{i-1}$, is a critical dependency of $v_i$, i.e., any delay in $v_{i-1}$ will result in delay in $v_i$.
  - The time for entire project depends on the time for the critical path.
  - Reducing the time of an off-critical-path task is of no help for reducing the total time for the project.
- **The Problems**
  - Find the critical path in a DAG
  - (Try to reduce the time for the critical path)

This is a precondition.

# Critical Path in a Task Graph

- **Earliest start time (est) for a task v**
  - If v has no dependencies, the est is 0
  - I v has dependencies, the est is the maximum of the earliest finish time of its dependencies.
- **Earliest finish time (eft) for a task v**
  - For any task: eft = est + duration
- **Critical path in a project is a sequence of tasks: $v_0$, $v_1$, …, $v_k$, satisfying:**
  - $v_0$ has no dependencies;
  - For any $v_i$=(i=1,2,…,k), $v_{i-1}$ is a dependency of $v_i$, such that est of $v_i$ equals eft of $v_{i-1}$;
  - eft of $v_k$, is maximum for all tasks in the project.

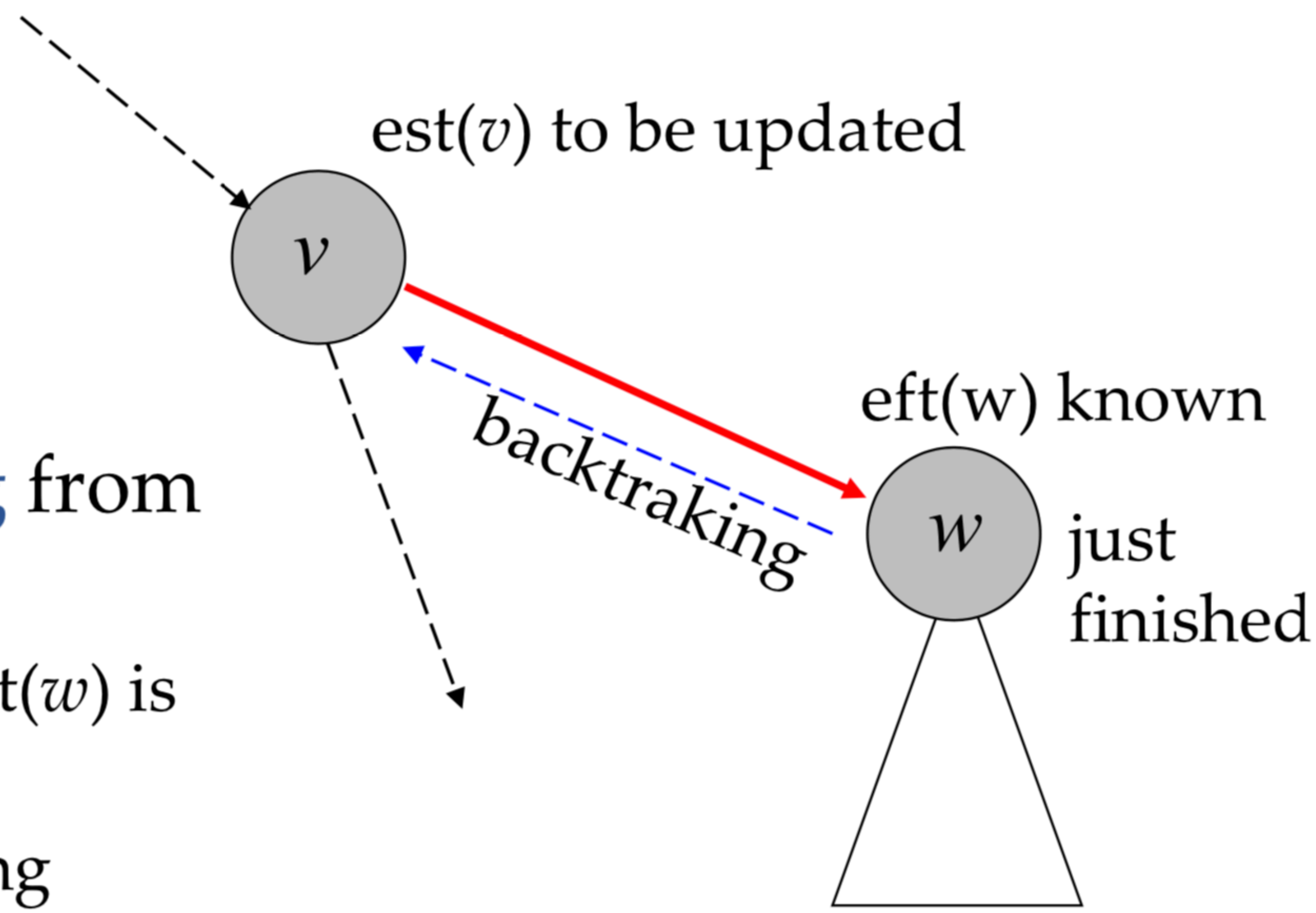
# DAG with Weights



# Critical Path Finding - DFS

- **Specialized parameters**
  - Array duration, keeps the execution time of each vertex.
  - Array critDep, keeps the critical dependency of each vertex.
  - Array eft, keeps the earliest finished time of each vertex.
- **Output**
  - Array topo, critDep, eft as filled.
- **Critical path is built by tracing the output.**

# Critical Path - Case 1



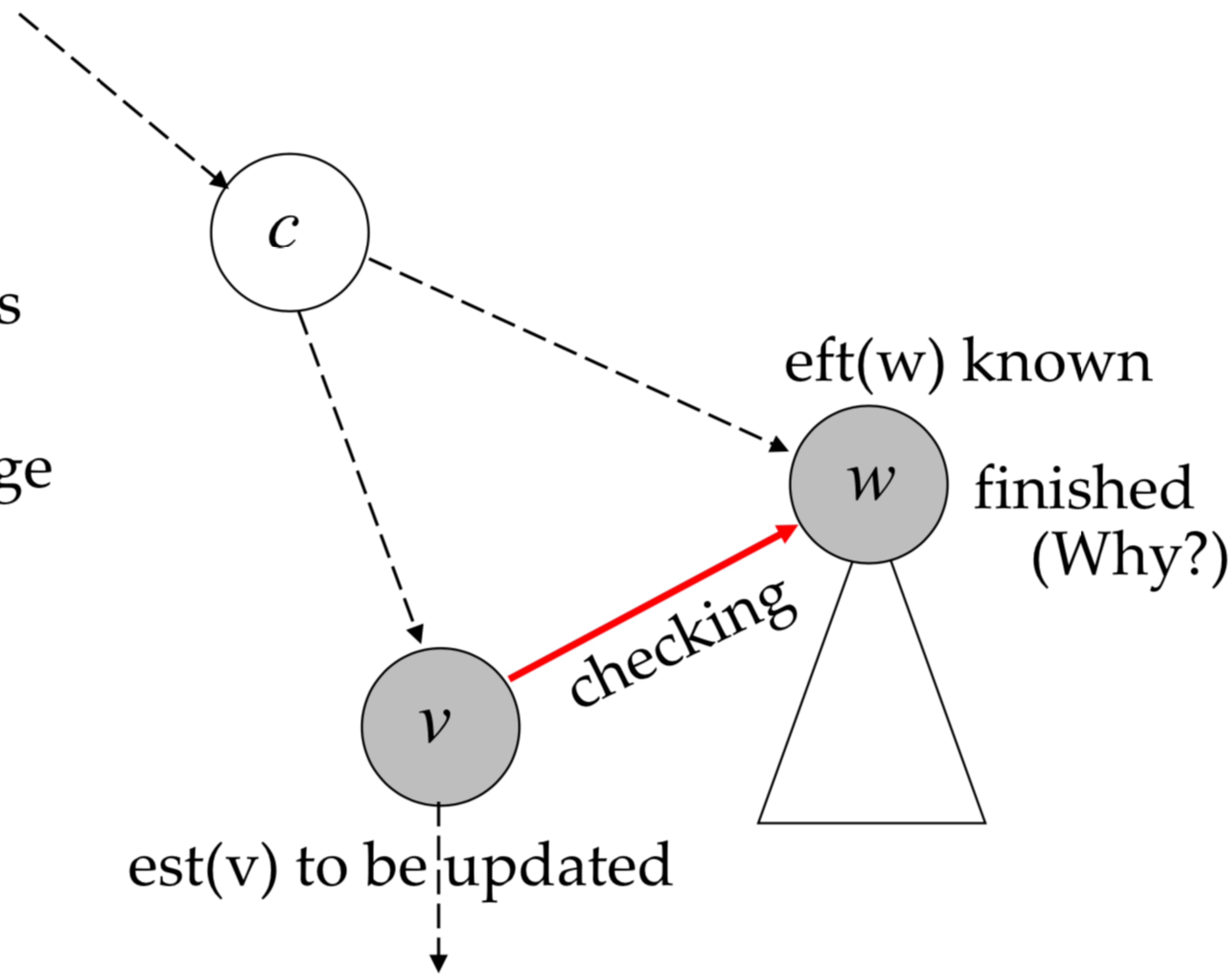

**Upon backtracking** from $w$:

- est($v$) is updated if eft($w$) is larger than est($v$)
- and the path including edge $vw$ is recognized as the critical path for tast $v$
- and the eft($v$) is updated accordingly

# Critical Path - Case 2

**Checking** $w$:

- est($v$) is updated if eft($w$) is larger than est($v$)
- and the path including edge $vw$ is recognized as the critical path for task $v$
- and the eft($v$) is updated accordingly



# Critical Path by DFS

- void dfsCritSweep(intList[] adjVertices, int n, int[] duration, int[] critDep, int[] eft)
- <Allocate color array and initialize to white>
- for each vertex v of G, in some order
- if(color[v]==white)
- dfsCrit(adjVertices, color, v, duration, critDep, eft);
- //continue loop
- return;

# Critical Path by DFS

- int dfsCrit(intList[] adjVertices, int[] color, int v, int[] duration int[] critDep, int eft)
- int w; intList remAdj; int est=0;
- color[v]=gray; critDep[v]=-1; remAdj=adjVertices[v];
- while(remAdj != nil) w=first(remAdj);
- if(color[w]==white)
- dfsCrit(adjVertices, color, w, duration, critDep, eft);
- if (eft[w]≥est) est=eft[w]; critDep[v]=w;
- else//checking for nontree edge
- if (eft[w]≥est) est=eft[w]; critDep[v]=w;
- remAdj=rest(remAdj);
- eft[v]=est+duration[v]; color[v]=black;
- return;

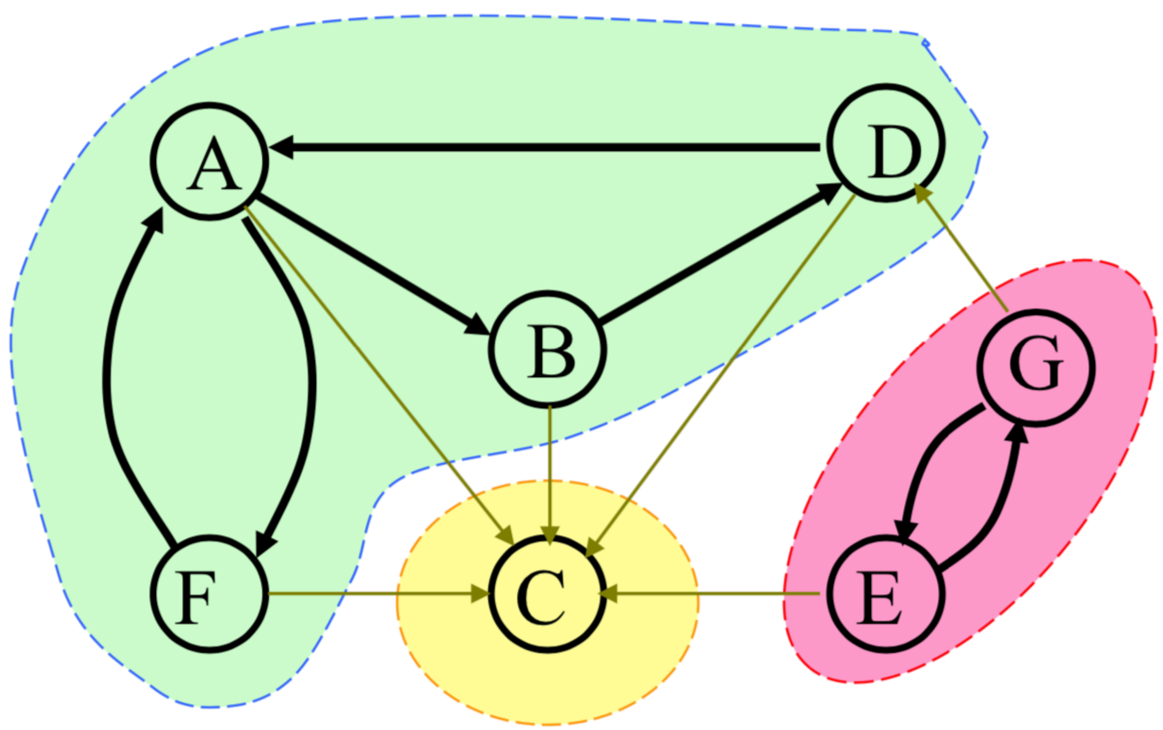
When is the eft[w] initialized?

Only black vertex

# Analysis of Critical Path Algorithm

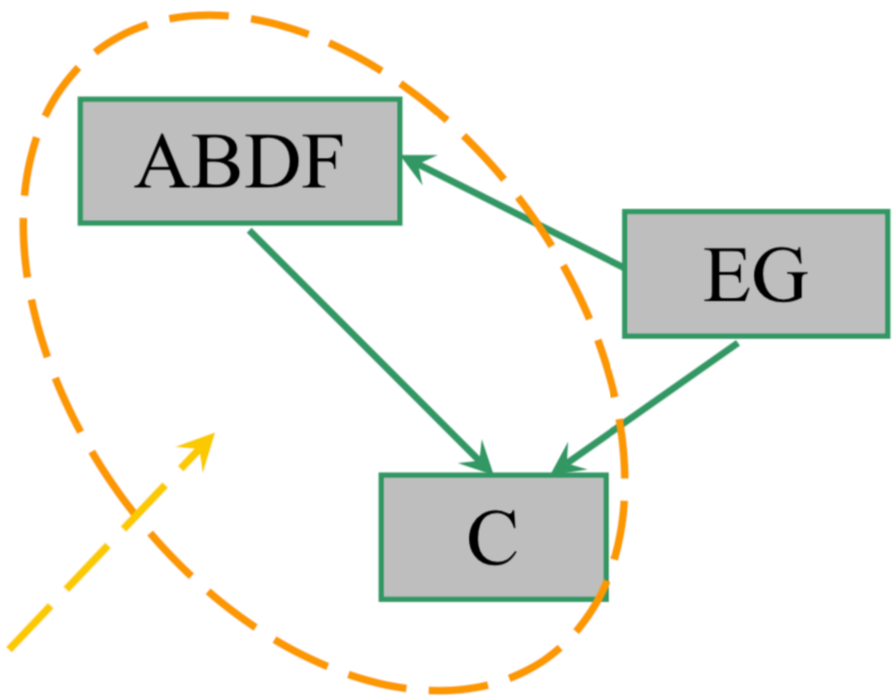
- **Correctness:**
  - When *eft*[*w*] is accessed in the while-loop, the w must not be gray (otherwise, there is a cycle), so, it must be black, with *eft* initialized.
  - According to DFS, each entry in the *eft* array is assigned a value exactly once. The value satisfies the definition of *eft*.
- **Complexity**
  - Simply same as DFS, that is $\Theta(n+m)$.

# SCC: Strongly Connected Component




**Graph G**
**3 Strongly Connected Components**

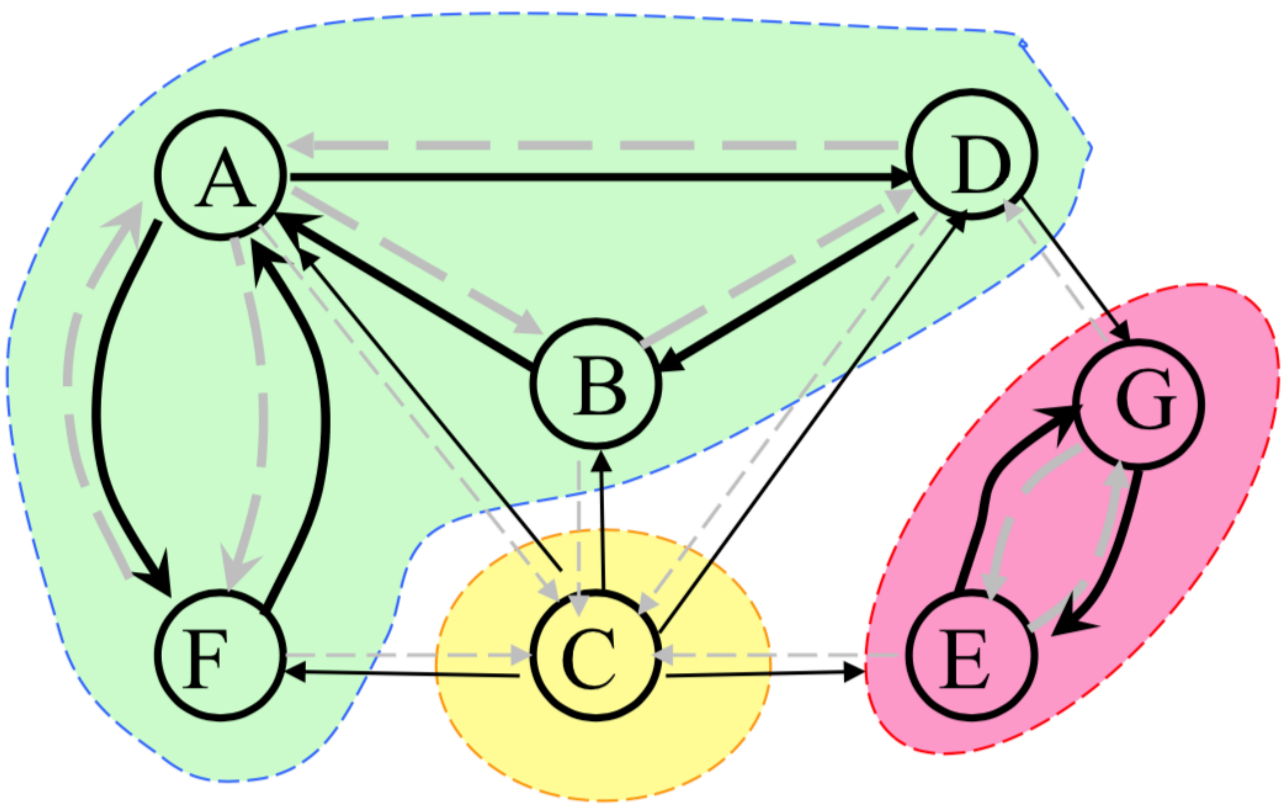
**Condensation Graph G↓**




It's acyclic, ***Why?***

Note: two SCC in one DFS tree

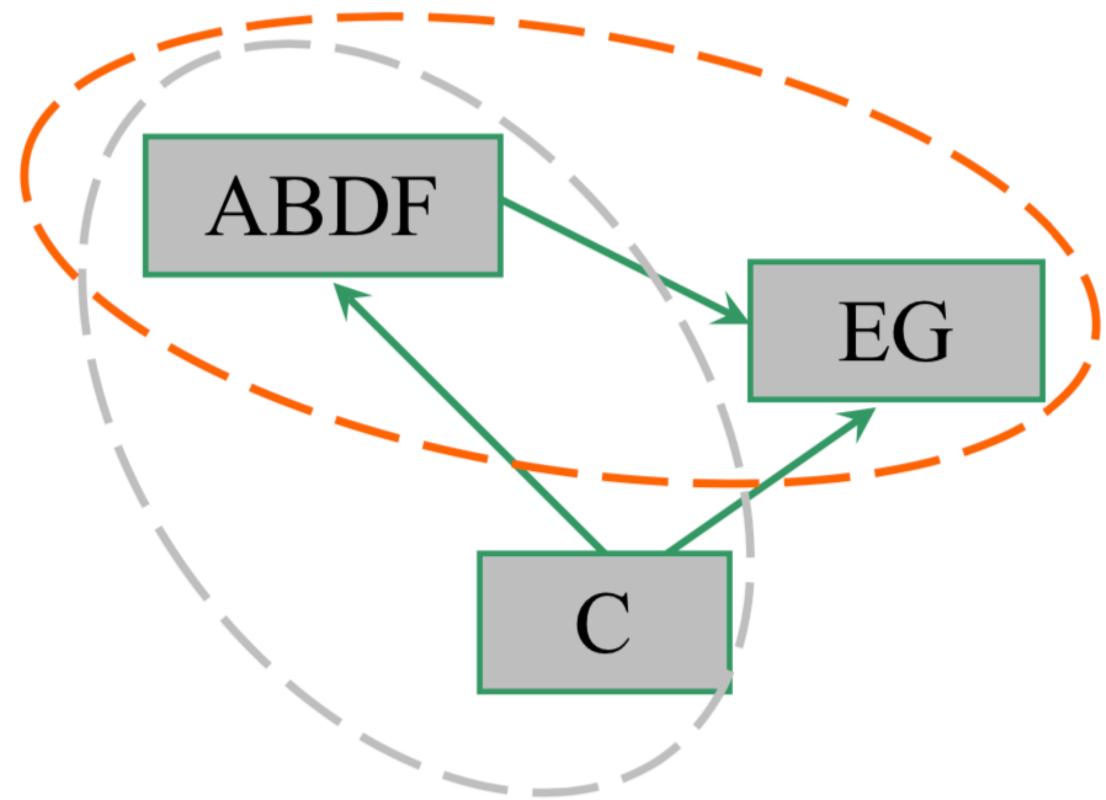
# Transpose Graph




**Tranpose Graph $G^T$**

**Connected Components unchanged according to vertices**

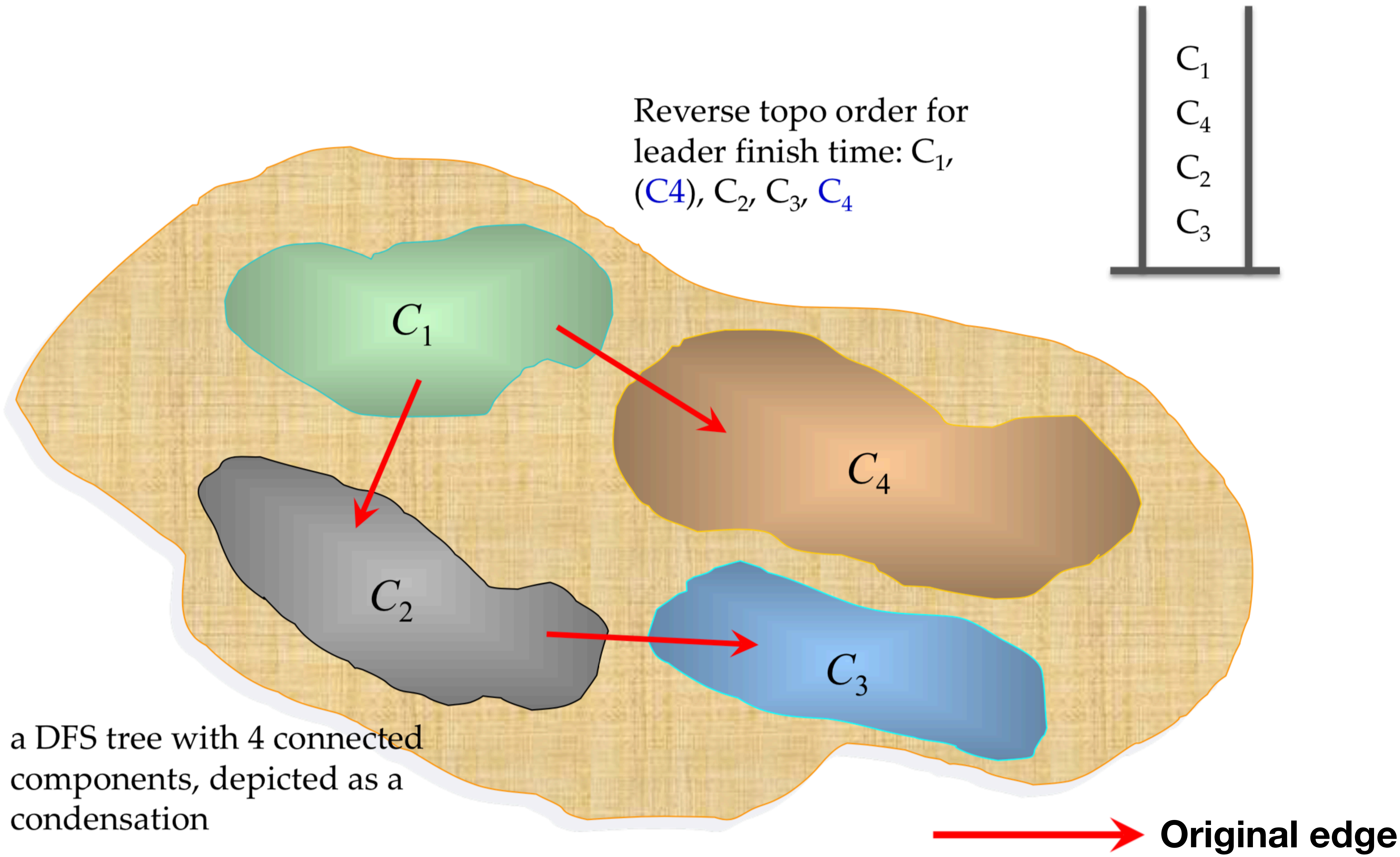
**Condensation Graph G↓**




**But, DFS tree changed**

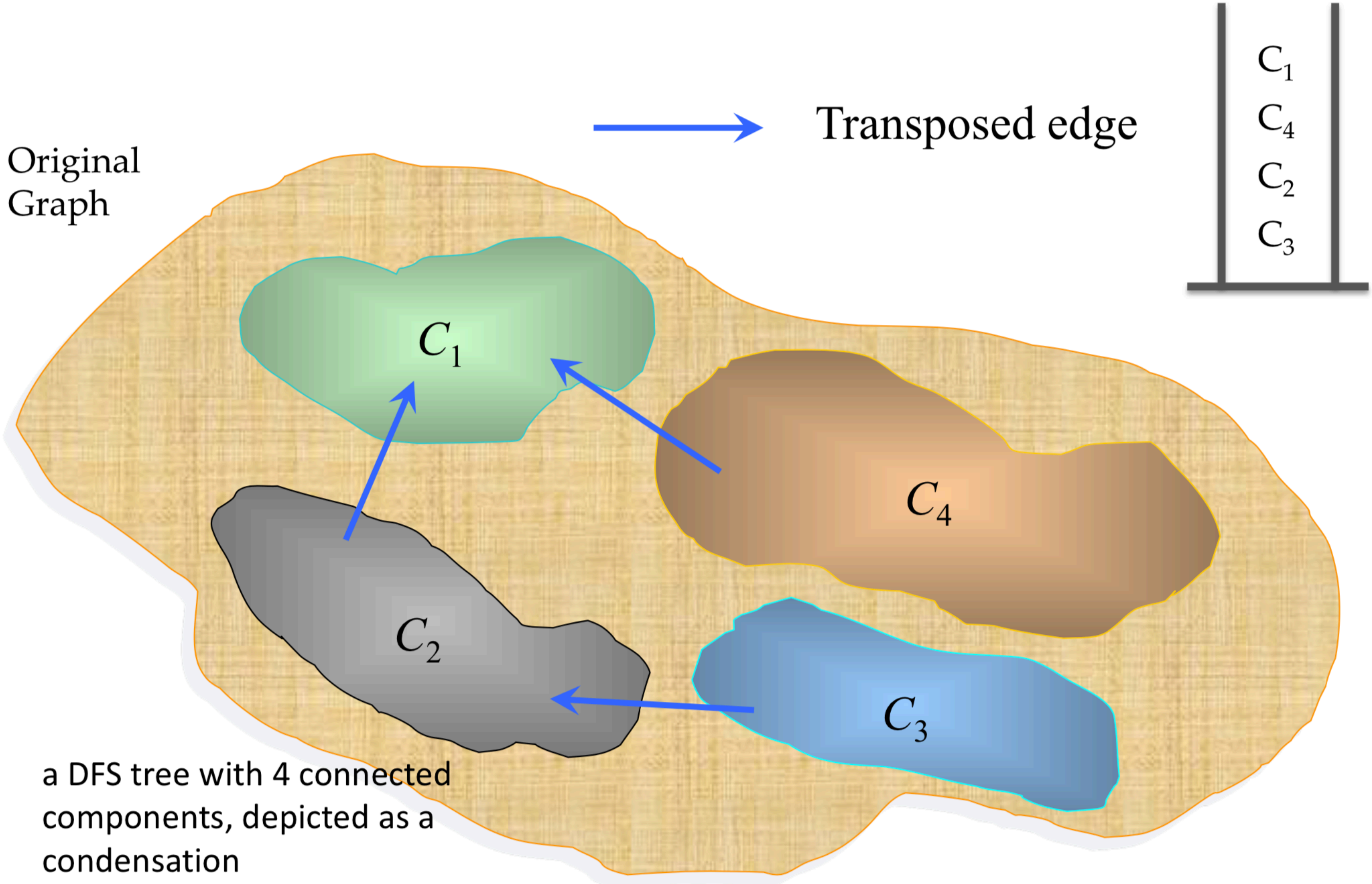
# Basic Idea - G

Reverse topo order for leader finish time: $C_1$, (C4), $C_2$, $C_3$, $C_4$

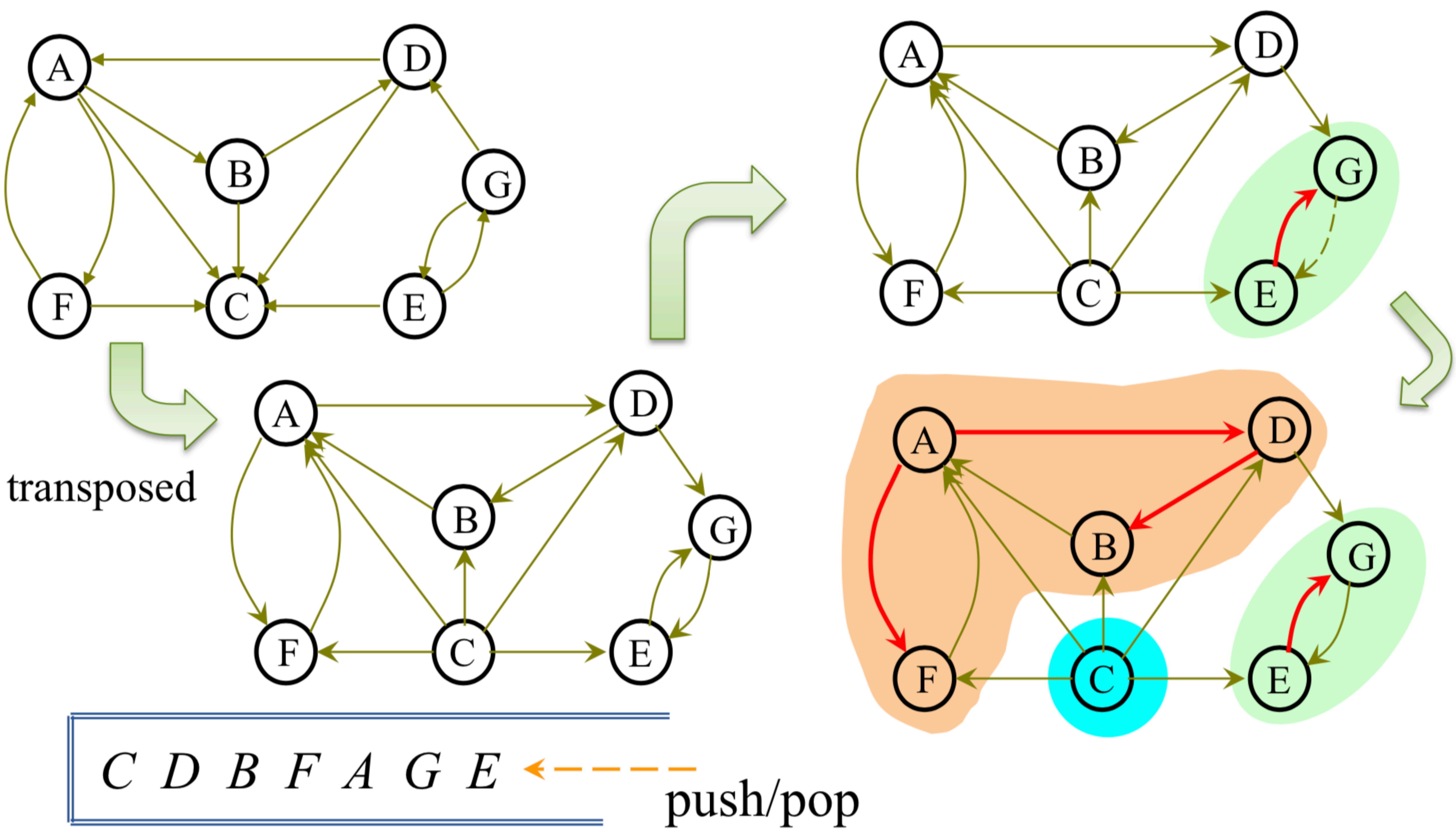



a DFS tree with 4 connected components, depicted as a condensation

# Basic Idea - $G^T$



# SCC - An Example



# Strong Component Algorithm: Outline

- void strongComponents(intList[] adjvertices, int n, int[] scc)
- //Phase 1
- 1. intStack finishStack=create(n);
- 2. perform a depth-first search on G, using the DFS skeleton. At postorder processing for vertex v, insert the statement: push(finishStack, v)
- //Phase 2
- 3. Compute $G^T$, the transpose graph, represented as array adjTrans of adjacency list.
- 4. dfsTsweep(adjTrans, n, finishStack, scc);
- return

Note: G and $G^T$ have the same SCC sets

# Strong Component Algorithm: Core

- void dfsTSweep(intList[] adjVertices, int n, intStack finishStack, int[] scc)
-     <Allocate color array and initialize to white>
-     while(finishStack is not empty)
-         int v=top(finishStack);
-         pop(finishStack);
-         if(color[v]==white)
-             dfsT(adjVertices, color, v, v, scc);
-     return;
- void dfsT(intList[] adjTrans, int[] color, int v, int leader, int[] scc)
-     Use the standard depth-first search skeleton. At postorder processing for vertex v insert the statement:
-         scc[v]=leader;
-         Pass leader and scc into recursive calls.

# Leader of a Strong Component

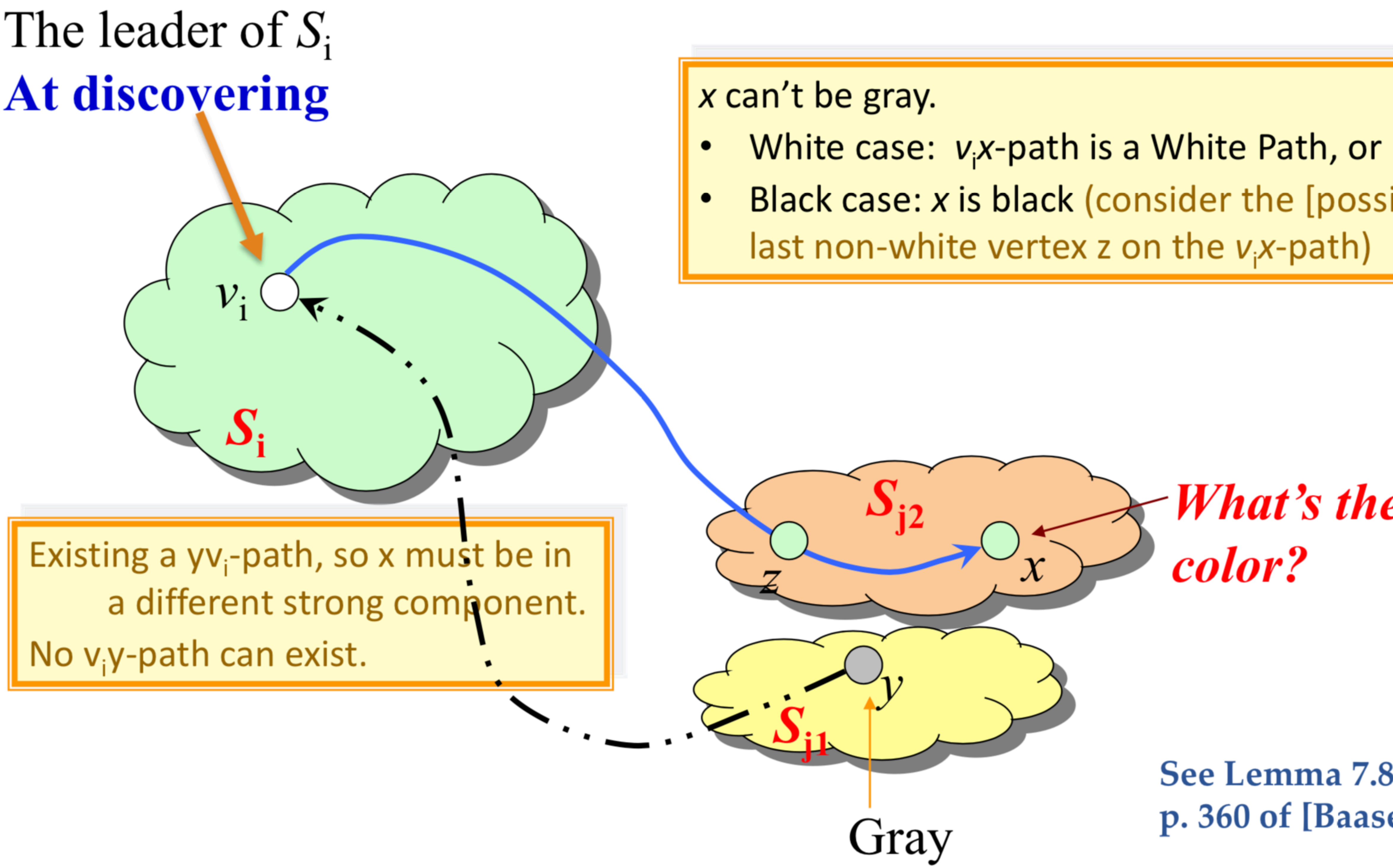
- **For a DFS, the first vertex discovered in a strong component $S_i$ is called the leader of $S_i$.**

- **Each DFS tree of a digraph G contains only complete strong components of G, one or more.**

  - Proof: Applying White Path Theorem whenever the leader of $S_i$ ($i$=1,2,...$p$) is discovered, starting with all vertices being white.

- **The leader of $S_i$ is the last vertex to finish among all vertices of $S_i$. (since all of them in the same DFS tree)**

# Path between SCCs




*x* can't be gray.

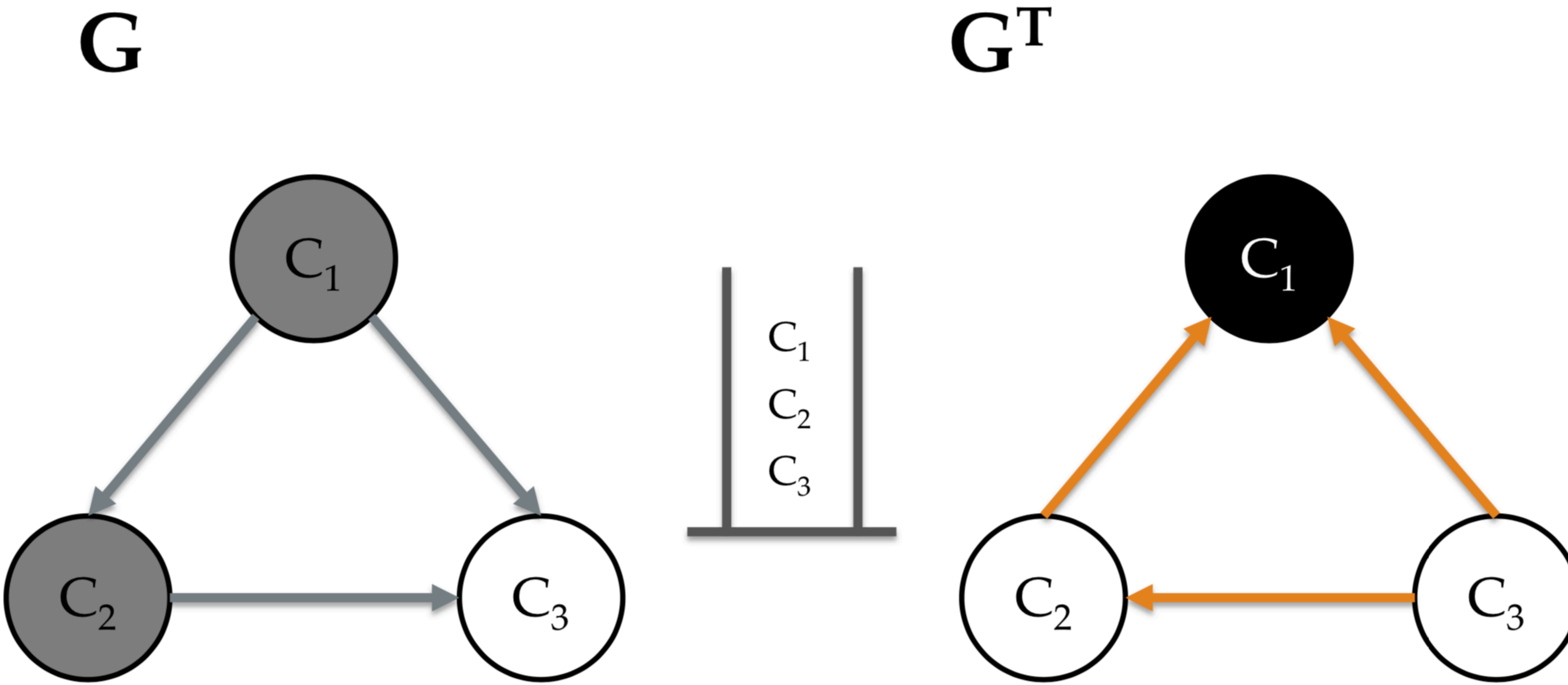
- White case: $v_i x$-path is a White Path, or
- Black case: *x* is black (consider the [possible] last non-white vertex z on the $v_i x$-path)

Existing a $yv_i$-path, so x must be in a different strong component.
No $v_i y$-path can exist.

See Lemma 7.8 & 7.9 p. 360 of [Baase01]

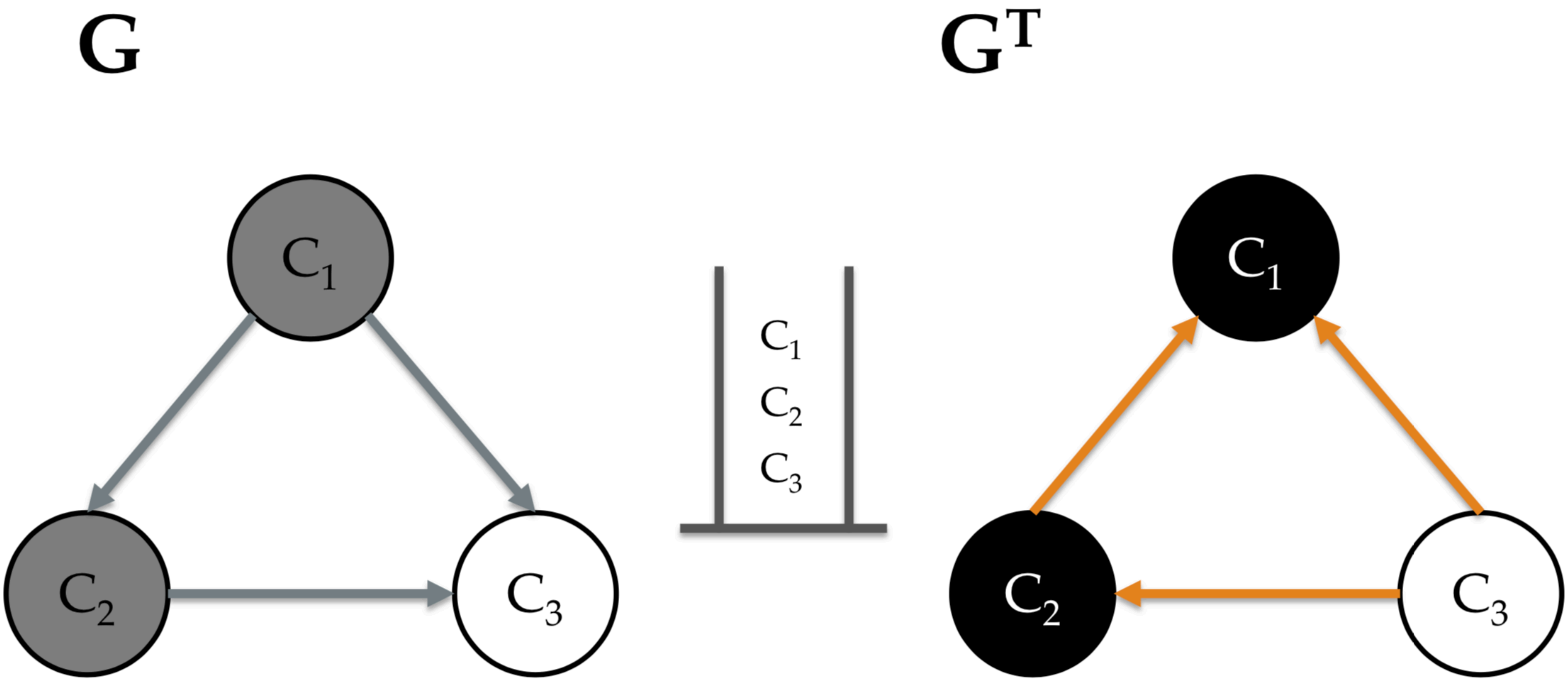
# $C_1$: The End Case

Looking at $C_2$, $C_3$ from $C_1$



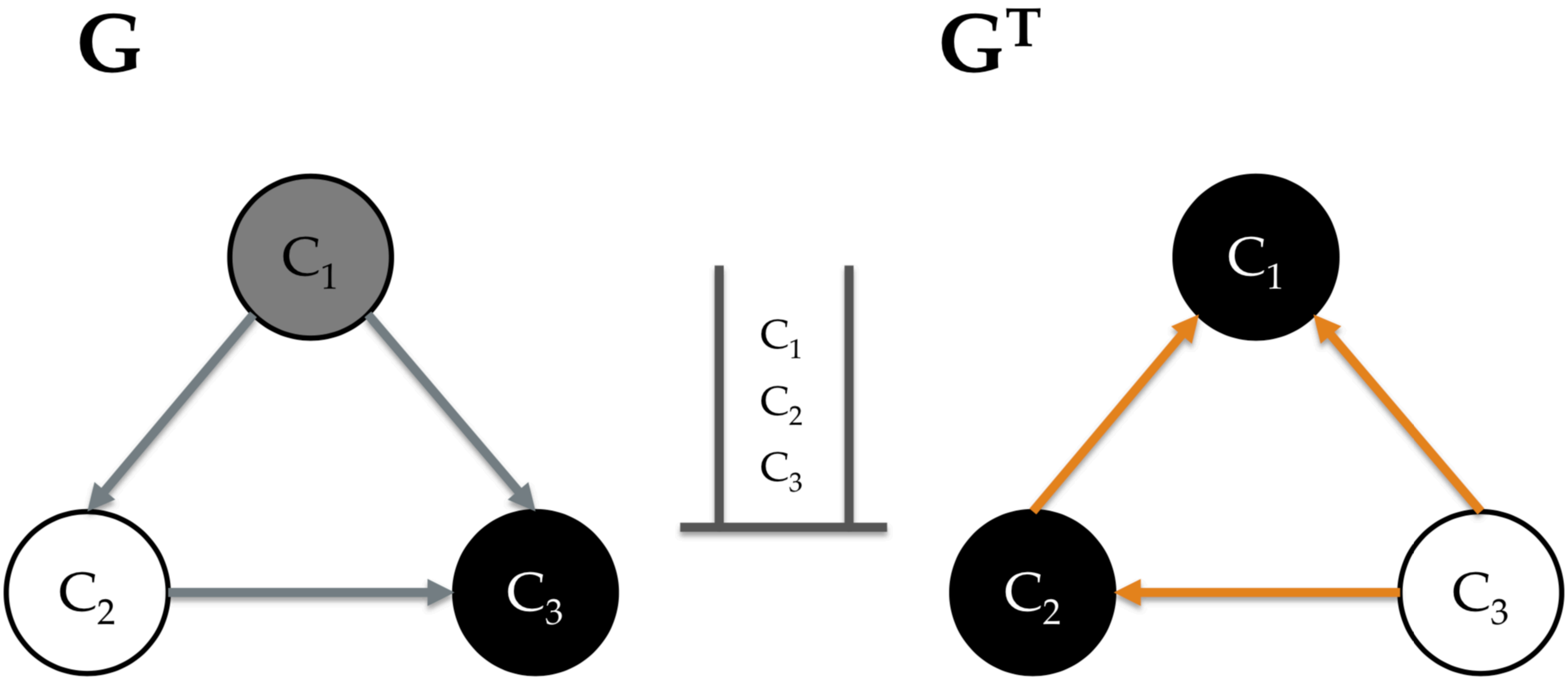
# $C_2$: The White Case

Looking at $C_3$ from $C_2$



# $C_2$: The Black Case

Looking at $C_3$ from $C_2$



# Active Intervals

- **If there is an edge from $S_i$ to $S_j$, then it is impossible that the active interval of $v_j$ is entirely after that of $v_i$.** **(Note: for leader $v_i$ only)**

    - There is no path from a leader of a strong component to any gray vertex.

    - If there is a path from the leader $v$ of a strong component to any $x$ in a different strong component, $v$ finishes later than $x$.

# Correctness of Strong Component Algorithm (1)

- **In phase 2, each time a white vertex is popped from *finishStack*, that vertex is the Phase 1 leader of a strong component.**
  - The later finished, the earlier popped
  - The leader is the first to get popped in the strong component it belongs to
  - If x popped is not a leader, then some other vertex in **the** strong component has been visited previously. But not a partial strong component can be in a DFS tree, so, x must be in a completed DFS tree, and is not white.

# Correctness of Strong Component Algorithm (2)

- **In phase 2, each depth-first search tree contains exactly one strong component of vertices**
  - Only “exactly one” need to be proved
  - Assume that $v_i$, a phase 1 leader is popped. If another component $S_j$ is reachable from $v_i$ in $G^T$, there is a path in $G$ from $v_j$ to $v_i$. So, in phase 1, $v_j$ finished later than $v_i$, and popped earlier than $v_i$ in phase 2. So, when $v_i$ popped, all vertices in $S_j$ are black. So, $S_j$ are not contained in DFS tree containing $v_i(S_i)$.

Thank you!
Q & A