Introduction to

Algorithm Design and Analysis

[02] Asymptotics

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In the Last Class…

- **Algorithm - the spirit of computing**
  - Model of computation
- **Algorithm design and analysis**
  - Design
    - Correctness proof by induction
  - Analysis
    - Worst-case / average-case complexity

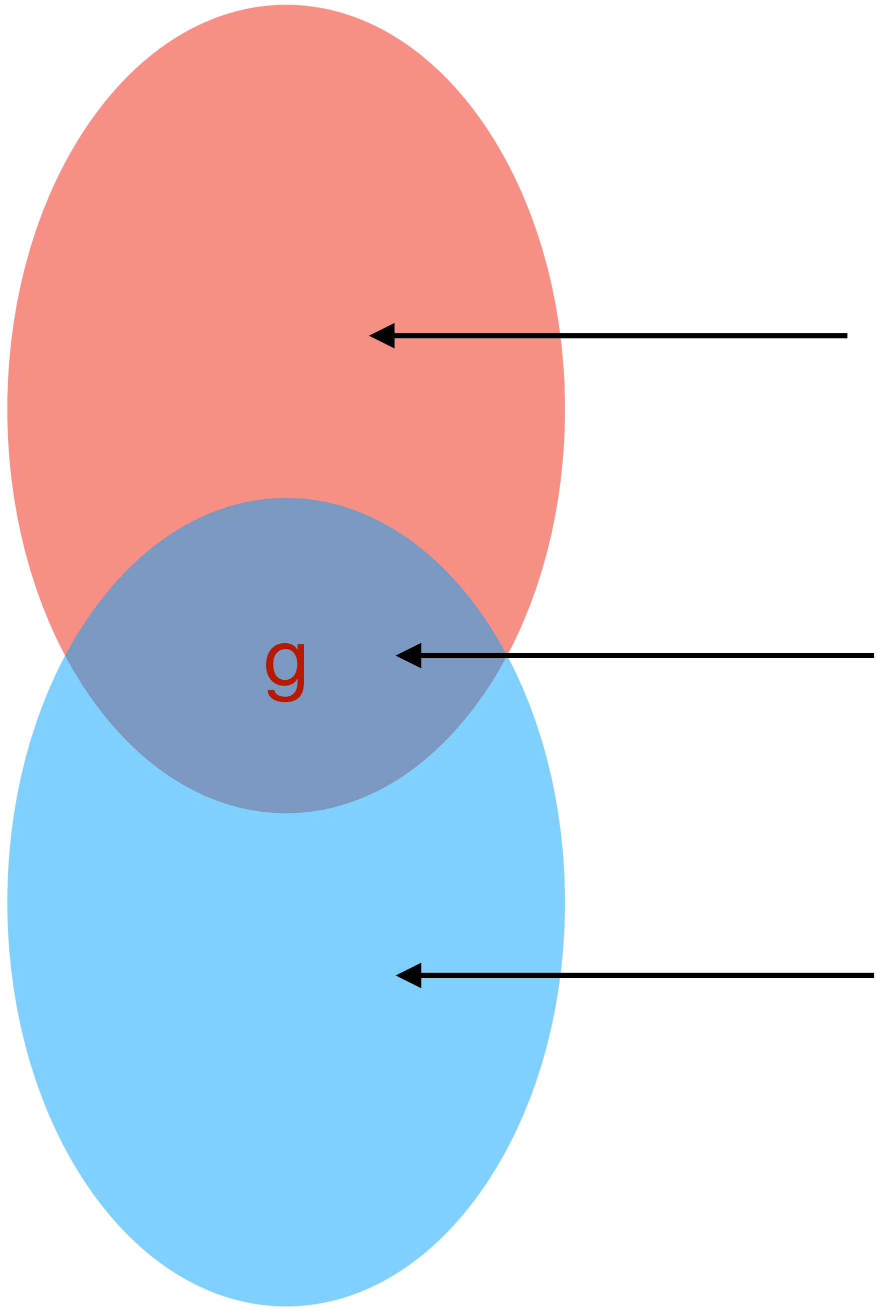
Asymptotic Behavior

- **Asymptotic growth rate of functions**
  - Basic idea
- **Key notations**
  - O, Ω, Θ
  - o, ω
- **Brute force enumeration**
  - By iteration
  - By recursion

How to Compare Two Algorithms

- **Algorithm analysis, with simplifications**
  - Measuring the cost by the number of critical operations
  - Large input size only
    - Only the leading term in f(n) is considered
    - Constant coefficients are ignored
- **Capturing the essential part in the cost in a mathematical way**
  - Asymptotic growth rate of f(n)

Relative Growth Rate



"Big Oh"

- **Basic idea $f(n) \in O(g(n))$**
  - For sufficiently large input size, g(n) is an upper bound for f(n)
- **Definition - "ε-N"**
  - Giving $g: N \to R^+$, then O(g) is the set of $f: N \to R^+$, such that for some $c \in R^+$ and some $n_0 \in N$, $f(n) \le cg(n)$ for all $n \ge n_0$
- **Definition - "$\lim_{n \to \infty}$"**
  - $f \in O(g)$ if $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c < \infty$$

The limit may not exist, though it usually does.

Example

- **Let f(n)=n², g(n)=nlogn, then:**
  - f∉O(g), since

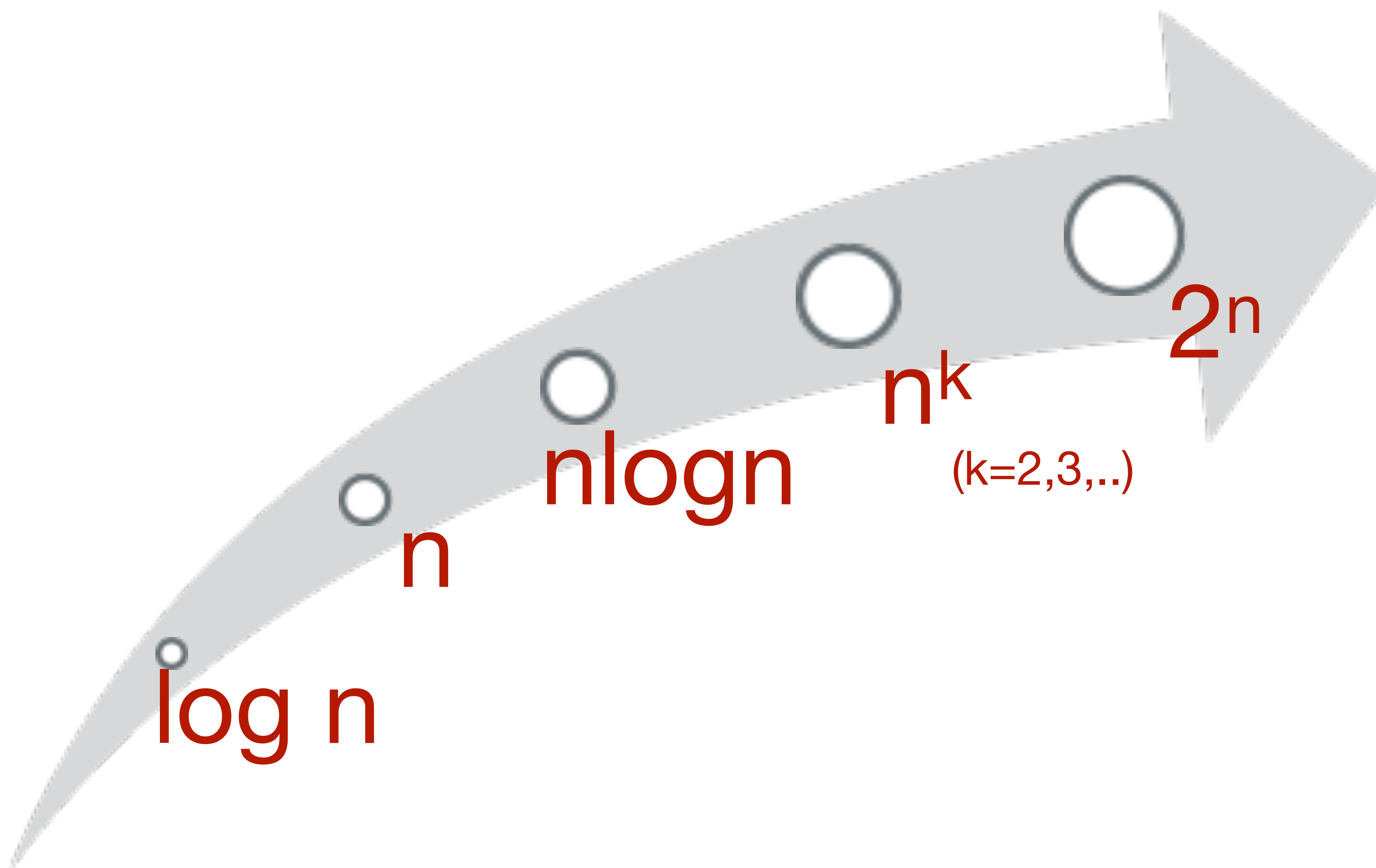
$$\lim_{n\to\infty}\frac{n^2}{nlogn}=\lim_{n\to\infty}\frac{n}{logn}=\lim_{n\to\infty}\frac{1}{\frac{1}{nln2}}=+\infty$$

L'Hospital's rule

  - g∈O(f), since

$$\lim_{n\to\infty}\frac{nlogn}{n^2}=\lim_{n\to\infty}\frac{logn}{n}=\lim_{n\to\infty}\frac{1}{nln2}=0$$

Asymptotic Growth Rate



Asymptotic Order

- **Logarithm logn**

$$\log n \in O(n^{\alpha})$$ for any α>0

- **Power n^k**

$$n^k \in O(c^n)$$ for any c>1

- **Factorial n!**

$$n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$$ Stirling's formula

"Big Ω"

- **Basic idea $f(n) \in \Omega(g(n))$**
  - Dual of "O"
- **Definition - "ε-N"**
  - Giving $g: N \rightarrow R^+$, then $\Omega(g)$ is the set of $f: N \rightarrow R^+$, such that for some $c \in R^+$ and some $n_0 \in N$, $f(n) \geq cg(n)$ for all $n \geq n_0$
- **Definition - "$\lim_{n\rightarrow\infty}$"**
  - $f \in \Omega(g)$ if $$\lim_{n\rightarrow\infty} \frac{f(n)}{g(n)} = c > 0$$ the limit may be ∞

The Set Θ

- **Basic idea $f(n) \in \Theta(g(n))$**
  - Roughly the same
  - $\Theta(g) = O(g) \cap \Omega(g)$
- **Definition - "ε-N"**
  - Giving $g: N \to R^+$, then $\Theta(g)$ is the set of $f: N \to R^+$, such that for some $c_1, c_2 \in R^+$ and some $n_0 \in N$, $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$, for all $n \ge n_0$
- **Definition - "$\lim_{n\to\infty}$"**
  - $f(n) \in \Theta(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c (0 < c < \infty)$

Some Empirical Data

algorithm		1	2	3	4
Run time in *ns*		$1.3n^3$	$10n^2$	$47n\log n$	$48n$
time for size	10^3	1.3*s*	10*ms*	0.4*ms*	0.05*ms*
	10^4	22*m*	1*s*	6*ms*	0.5*ms*
	10^5	15*d*	1.7*m*	78*ms*	5*ms*
	10^6	41*yrs*	2.8*hrs*	0.94*s*	48ms
	10^7	41*mill*	1.7*wks*	11*s*	0.48*s*
max Size in time	sec	920	10,000	1.0×10^6	2.1×10^7
	min	3,600	77,000	4.9×10^7	1.3×10^9
	hr	14,000	6.0×10^5	2.4×10^9	7.6×10^{10}
	day	41,000	2.9×10^6	5.0×10^{10}	1.8×10^{12}
time for 10 times size		×1000	×100	×10+	×10

on 400Mhz Pentium II, in C

from: Jon Bentley: *Programming Pearls*

Properties of O, Ω and Θ

- **Transitive property**
  - if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$
- **Symmetric properties**
  - $f \in O(g)$ if and only if $g \in \Omega(f)$
  - $f \in \Theta(g)$ if and only if $g \in \Theta(f)$
- **Order of sum function**
  - $O(f+g)=O(\max(f,g))$

“Little Oh”

- **Basic idea $f(n) \in o(g(n))$**
  - Non-ignorable gap between f and its upper bound g
- **Definition - “ε-N”**
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- **Definition - “$\lim_{n\to\infty}$”**
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"Little ω"

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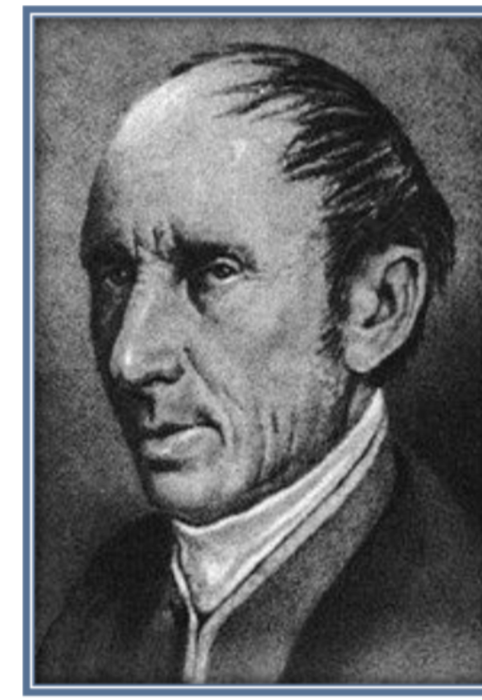
Do You Know Infinity

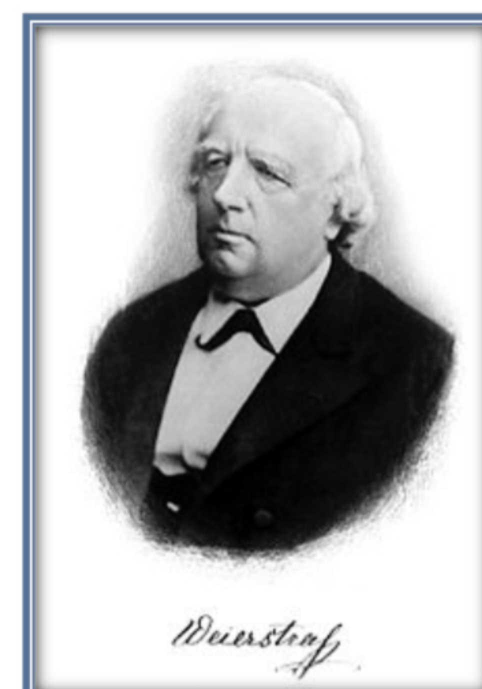
- **Mathematical analysis**
  - Firm foundation

Cauchy



- **How to talk about infinity?**
  - (ε-N)-definition
  - (ε-δ)-definition

Weierstrass



Brute Force Enumeration by Iteration

- **Swapping array elements**
  - <time, space>
    - From <$O(n^2)$, $O(1)$>
    - To <$O(n)$, $O(n)$>
    - To <$O(n)$, $O(1)$>
- **Maximum subsequence sum**
  - Time
    - From $O(n^3)$
    - To $O(n^2)$
    - To $O(n\log n)$
    - To $O(n)$

Swapping Array Elements

- E.g., 1,2,3,4 | 5,6,7 => 5,6,7,1,2,3,4
- Brute force

Space sensitive

	Time	Space
BF1	$O(n^2)$	$O(1)$
BF2	$O(n)$	$O(n)$
Your Task	$O(n)$	$O(1)$

Time sensitive

- Your task
  - Both time and space efficient

Max-sum subsequence

- **The problem: Given a sequence S of integer, find the largest sum of a consecutive subsequence of S, (0, if all negative items)**

An example: 2, 11, -4, 13, -5, -2; the result 22: (2, 11, -4, 13)

A brute-force algorithm:

```
MaxSum = 0;
for(i = 0; i < N; i++)
  for(j = i; j < N; j++){
      ThisSum = 0;
      for(k = i; k <= j; k++)
          ThisSum += A[k];
      if(ThisSum > MaxSum)
          MaxSum = ThisSum;
  }
return MaxSum;
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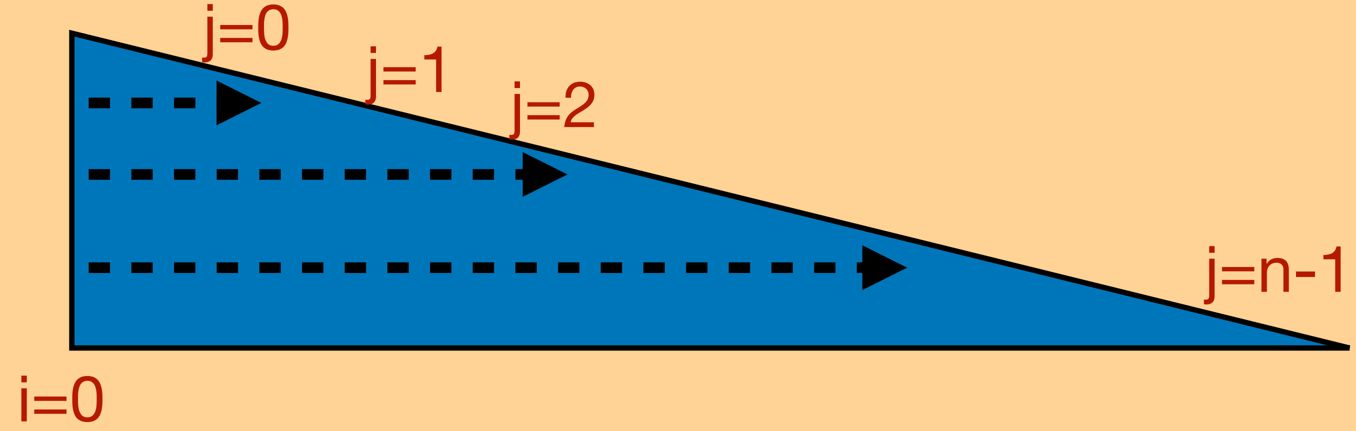

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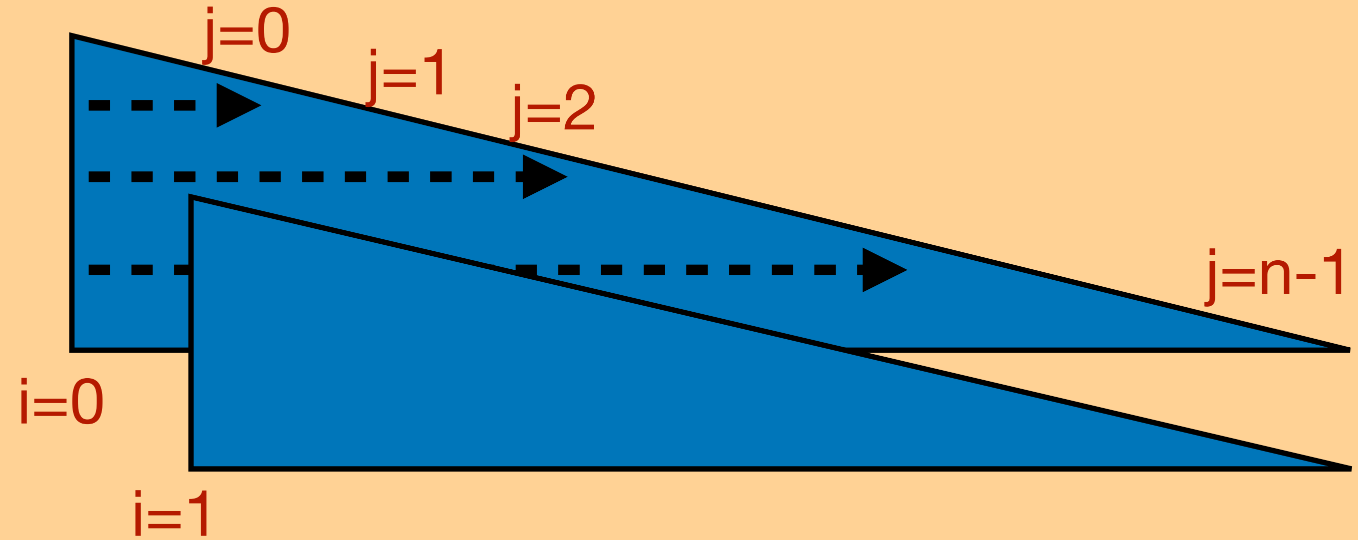

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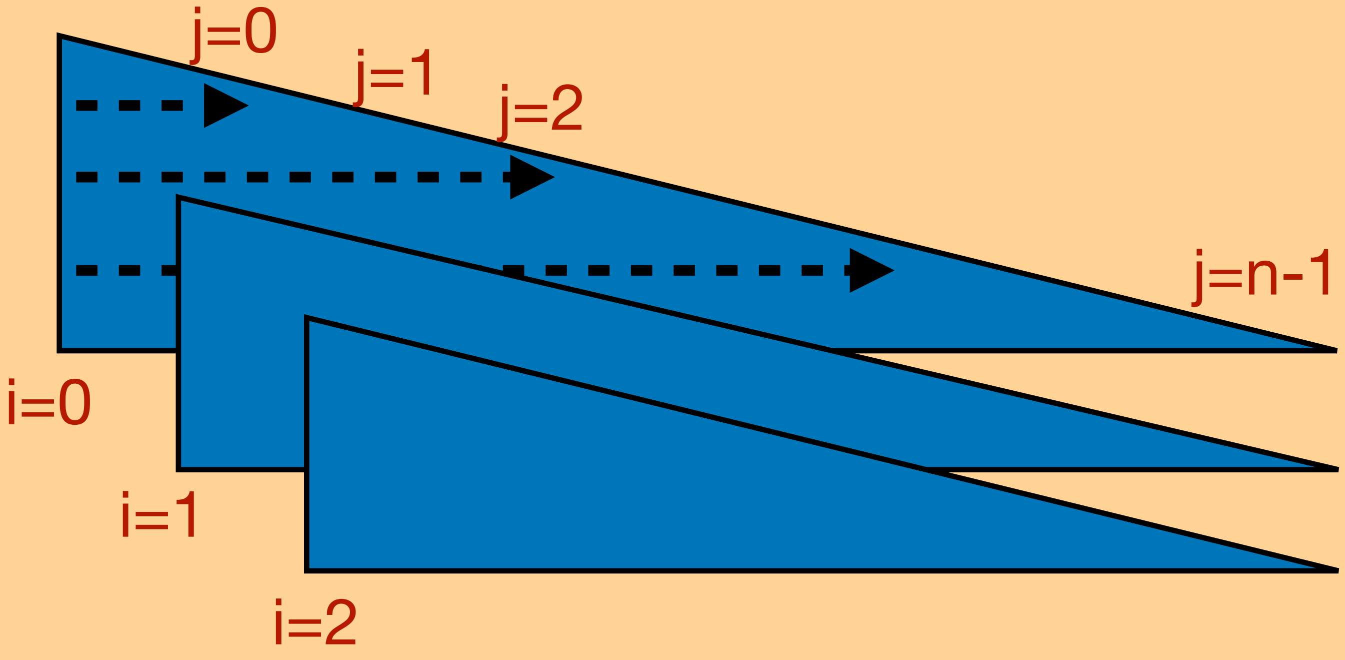

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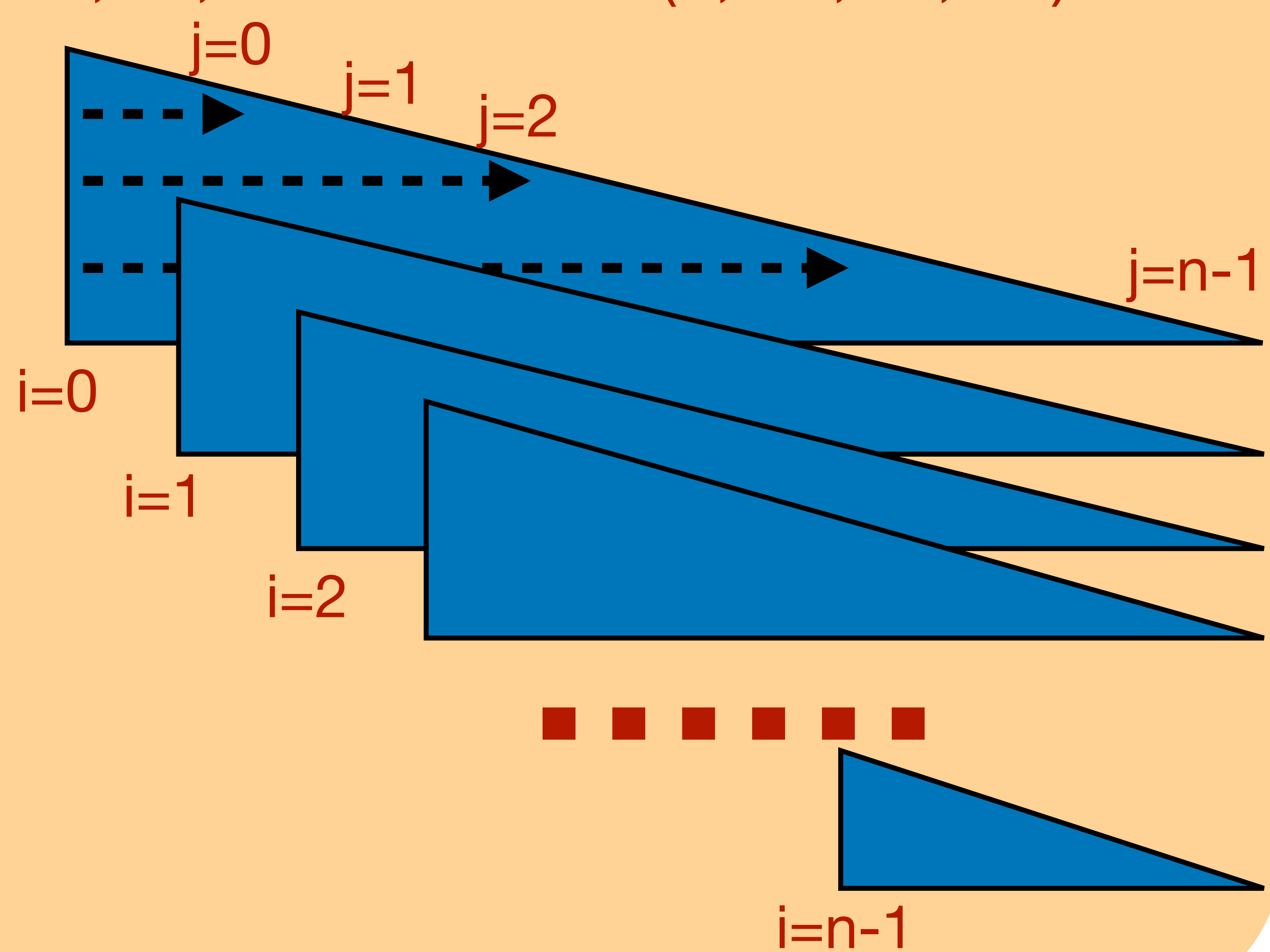

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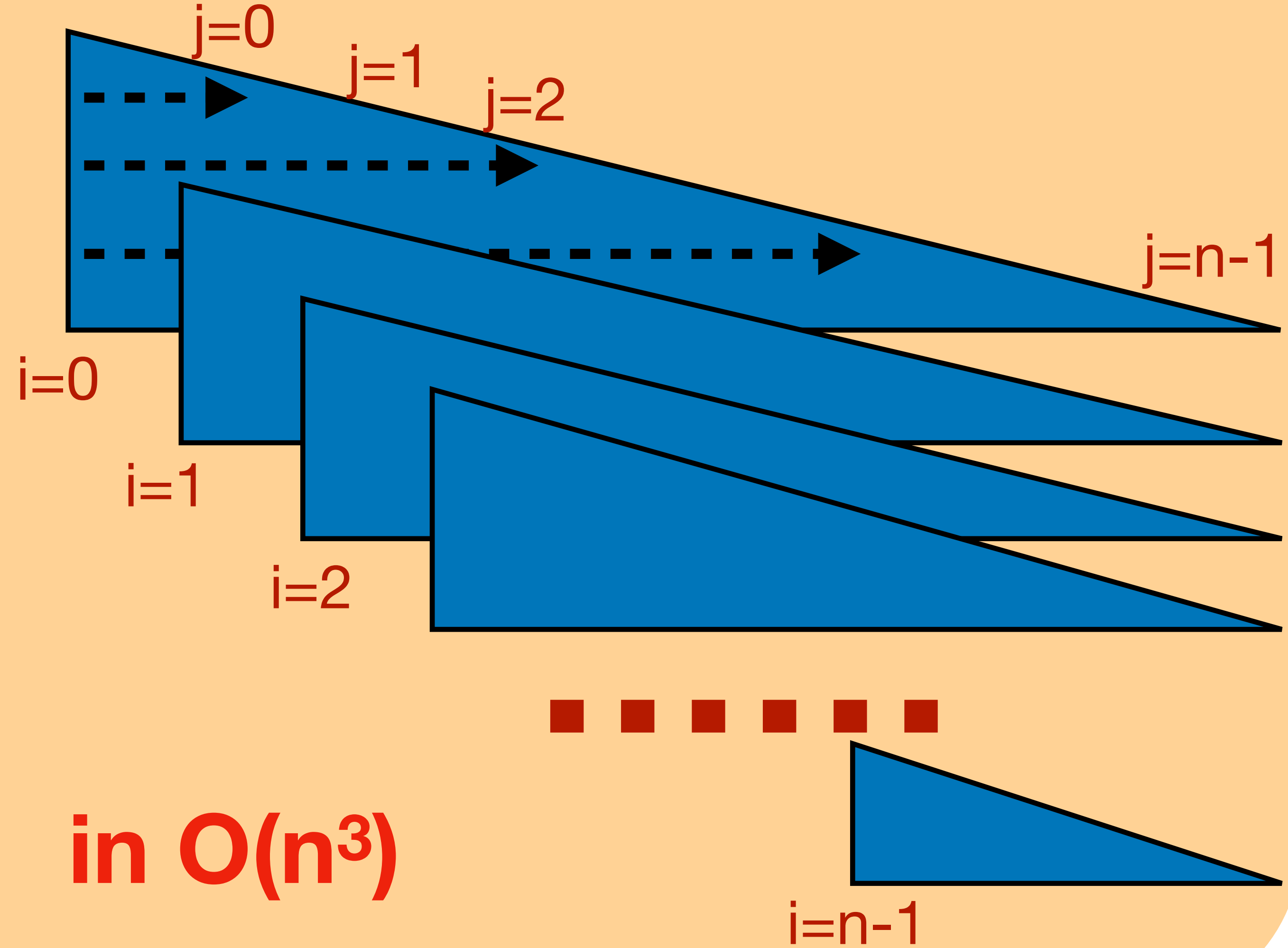

in O(n^3)

More Precise Complexity

- **The total cost is:** $\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} \sum_{k=i}^{j} 1$

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$$\sum_{k=i}^{j} 1 = j - i + 1$$

More Precise Complexity

- **The total cost is:** $\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\sum_{k=i}^{j} 1$

$$\sum_{k=i}^{j} 1 = j - i + 1$$

$$\sum_{j=i}^{n-1} (j - i + 1) = 1 + 2 + ... + (n - i) = \frac{(n - i + 1)(n - i)}{2}$$

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$$\sum_{k=i}^{j} 1 = j - i + 1$$

$$\sum_{j=i}^{n-1} (j - i + 1) = 1 + 2 + \ldots + (n - i) = \frac{(n - i + 1)(n - i)}{2}$$

$$\sum_{i=0}^{n-1} \frac{(n - i + 1)(n - i)}{2} = \sum_{i=1}^{n} \frac{(n - i + 2)(n - i + 1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{n} i^2 - (n + \frac{3}{2}) \sum_{i=1}^{n} i + \frac{1}{2}(n^2 + 3n + 2) \sum_{i=1}^{n} 1$$

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$$\sum_{j=i}^{n-1} (j - i + 1) = 1 + 2 + \ldots + (n - i) = \frac{(n-i+1)(n-i)}{2}$$

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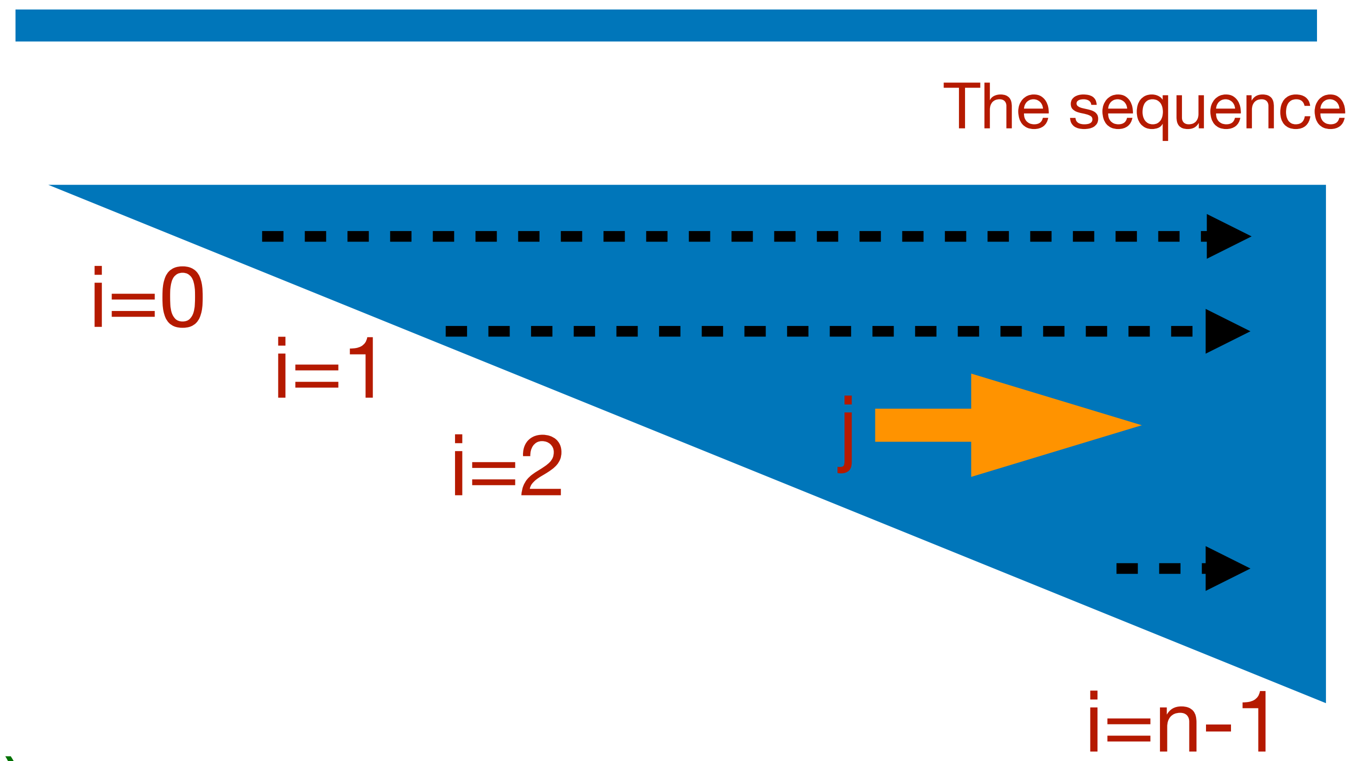
$$= \frac{1}{2} \sum_{i=1}^{n} i^2 - (n + \frac{3}{2}) \sum_{i=1}^{n} i + \frac{1}{2}(n^2 + 3n + 2) \sum_{i=1}^{n} 1$$

$$= \frac{n^3 + 3n^2 + 2n}{6}$$

Decreasing the Number of Loops

A improved algorithm:

```
MaxSum = 0;
for(i = 0; i < N; i++){
   ThisSum = 0;
   for(j = i; j < N; j++){
        ThisSum += A[j];
        if(ThisSum > MaxSum)
           MaxSum = ThisSum;
   }
}
return MaxSum;
```



in $O(n^2)$

Power of Divide and Conquer

Power of Divide and Conquer

```
Center = (Left + Right) / 2;
MaxLeftSum = MaxSubSum(A, Left, Center); MaxRightSum = MaxSubSum(A, Center + 1, Right);

MaxLeftBorderSum = 0; LeftBorderSum = 0;

for (i = Center; i >= Left; i--)

{
  LeftBorderSum += A[i];
  if (LeftBorderSum > MaxLeftBorderSum) MaxLeftBorderSum = LeftBorderSum;
}

MaxRightBorderSum = 0; RightBorderSum = 0;

for (i = Center + 1; i <= Right; i++)

{
  RightBorderSum += A[i];
  if (RightBorderSum > MaxRightBorderSum) MaxRightBorderSum = RightBorderSum;
}

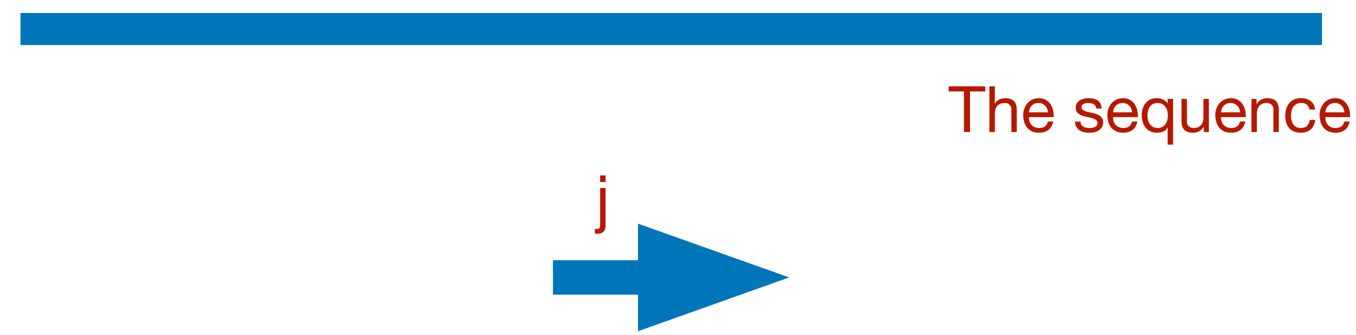
return Max3(MaxLeftSum, MaxRightSum, MaxLeftBorderSum + MaxRightBorderSum);
```

Note: this is the core part of the procedure, with base case and wrap omitted.

in O(nlogn)

A Linear Algorithm

```
ThisSum = MaxSum = 0;
for (j = 0; j < N; j++)
{
  ThisSum += A[j];
  if (ThisSum > MaxSum)
    MaxSum = ThisSum;
  else if (ThisSum < 0)
    ThisSum = 0;
}
return MaxSum;
```



This is an example of “online algorithm”

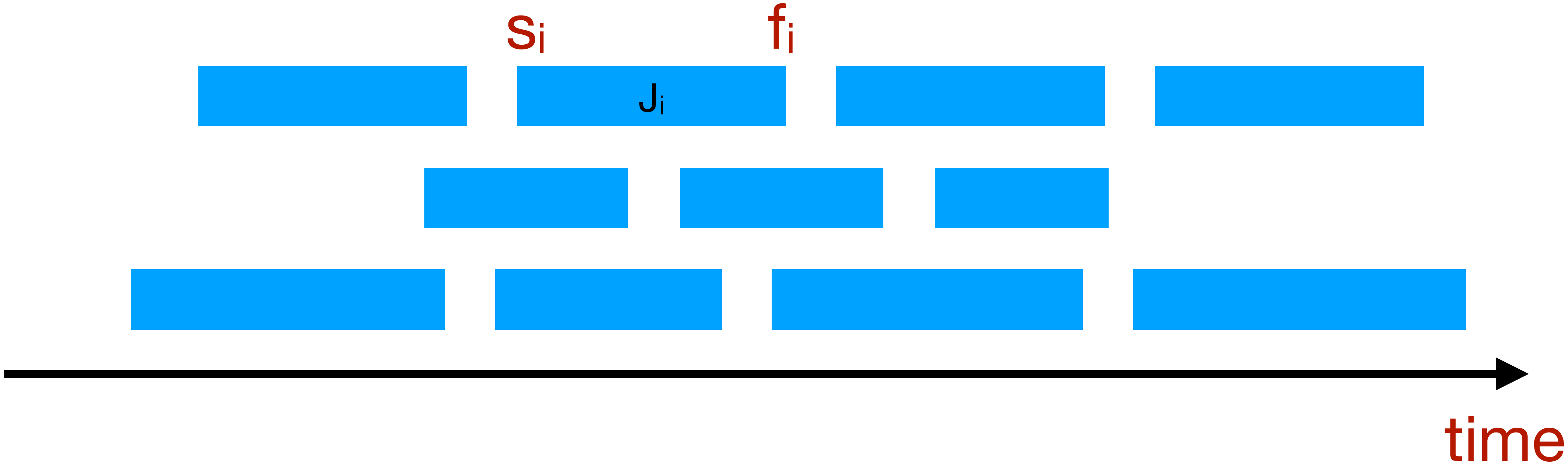
Negative item or subsequence cannot be a prefix of the subsequence we want.

Brute Force Enumeration By Recursion

- **Job scheduling**
  - Problem definition
  - Brute force recursion
  - Further improvements
- **Matrix chain multiplication**
  - Problem definition
  - Brute force recursion(s)
  - Further improvements

Job Scheduling

- **Jobs: $J_i = [s_i, f_i)$**
- **Max number of compatible jobs**
- **Further improvements**
  - Dynamic programming (L16)
  - Greedy algorithms (L14)



Matrix Chain Multiplication

- **The task:**
  - **Find the product:** $A_1 \times A_2 \times \ldots \times A_{n-1} \times A_n$
  - A_i **is 2-dimensional array of different legal size**
- **The Challenge:**
  - Matrix multiplication is associative
  - Different computing order results in great difference in the number of operations
- **The problem:**
  - Which is the best computing order

Cost of Matrix Multiplication

An example: $A_1 \times A_2 \times A_3 \times A_4$

30x1 1x40 40x10 10x25

$((A_1 \times A_2) \times A_3) \times A_4$: 20700 multiplications

$A_1 \times (A_2 \times (A_3 \times A_4))$: 11750

$(A_1 \times A_2) \times (A_3 \times A_4)$: 41200

$A_1 \times ((A_2 \times A_3) \times A_4)$: 1400

Solutions

- **Brute force recursion (L16)**
  - BF1
  - BF2
- **Dynamic programming (L16)**
  - Based on brute force recursion 2

Thank you!

Q & A